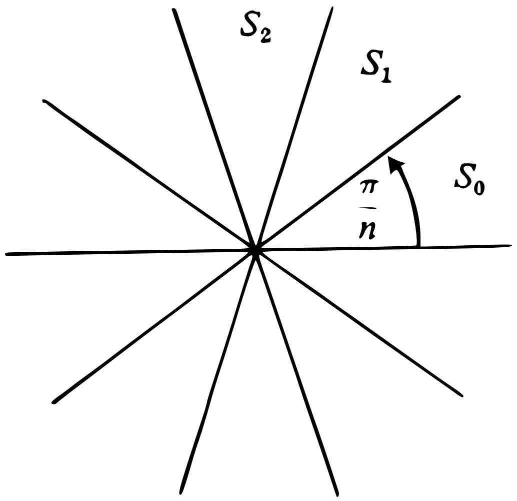
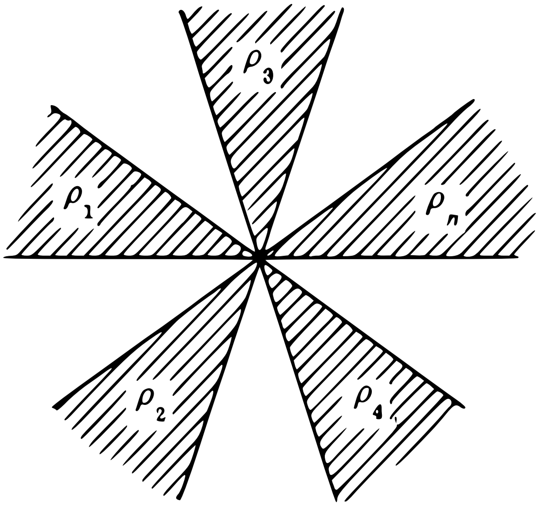
M.A. Naimark

LINEAR DIFFERENTIAL OPERATORS

Part 1

Elementary Theory of Linear Differential Operators





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PART I

Elementary Theory of Linear Differential Operators

WITH ADDITIONAL MATERIAL BY THE AUTHOR

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TRANSLATION EDITOR'S PREFACE

The first, Russian, edition of *Linear Differential Operators* by N. A. Naimark appeared in 1952. The English edition of this important book has had the benefit of advice from, and additional material supplied by, Professor Naimark, who has also written a preface for this new edition.

Some explanation may be called for in presenting at this time an English edition of this book. Since the original Russian edition, several presentations of the theory of linear ordinary differential operators have appeared, notably those of E. C. Titchmarsh (second edition, 1962), Coddington and Levinson (1955), K. S. Miller (1964), G. Hellwig (1964), and Dunford and Schwartz (1964). Some of these authors, notably Dunford and Schwartz, use methods of functional analysis; others, in particular, Titchmarsh, restrict themselves to classical analytical methods; none combines these two approaches so effectively as Naimark does in this book. There is no better introduction to the wide range of ideas and techniques required for the study of linear differential operators than that given here.

Part I presents a complete account of the so-called elementary theory of linear differential operators, in which the methods of functional analysis are kept in the background whilst use is made of the properties of functions of a complex variable. There is a detailed investigation of the general, regular, eigenvalue problem for ordinary differential equations with complex coefficients and regular boundary conditions. No comparable account exists elsewhere in the literature.

Part II of the book, which is to be published separately, begins with a compact account of the theory of linear operators in separable Hilbert spaces; these results are then used to present the properties of symmetric operators derived from ordinary differential expressions with real coefficients. This, in turn leads to an account of the spectral theory of such operators and the corresponding eigenfunction expansions in square-integrable function spaces. Next follows a long chapter on the determination of the deficiency indices and the spectrum of these differential operators. These results, whilst presented in the terminology of Hilbert space, are obtained by the use of methods of classical analysis, in particular, the development of certain asymptotic expansions for systems of differential equations. There is no comparable account of these expansions, which are largely due to Naimark himself, anywhere else in the literature. This second part of the book ends with an account of the work of Russian mathematicians on the inversion of the Sturm-Liouville problem.

The book has been expertly translated by Mr. E. R. Dawson, who has also brought the Bibliography up to date.

Mr. Dawson and I wish to express our thanks to several of our colleagues in Queen's College, Dundee, for their help in reading the manuscript and in checking the proofs.

W. N. EVERITT.

Queen's College, Dundee.
July 1966.

AUTHOR'S PREFACE TO THE ENGLISH EDITION

This edition incorporates the editorial amendments and additions which were made in the German edition of the book at the suggestion of the German editors, Professors H. O. Cordes and F. Rühs; the author takes this occasion to express his cordial acknowledgment to them.

In addition, a number of new changes and additions have been made in the English edition; these take into account some of the important advances in the theory of linear ordinary differential operators which have been made in the past few years. In this work the author has been assisted by Professor V. E. Lyantse, who has also written a Supplement to Part II. This Supplement deals with the author's work in the theory of non-selfadjoint singular differential operators, work which has later been extended by Professor Lyantse's own researches. It is the author's pleasant duty to express his deep gratitude to Professor Lyantse. The author also expresses his gratitude to Professor W. N. Everitt for undertaking the task of editing the English translation of the book.

M. A. NAIMARK.

Moscow, May 1966.

PREFACE TO THE RUSSIAN EDITION

Many topics in mathematical physics lead to the problem of determining the eigenvalues and eigenfunctions of differential operators and of expanding an arbitrary function as a series (or an integral) of eigenfunctions. A problem of this sort is encountered, for example, whenever the Fourier method is used to find a solution of a partial differential equation under prescribed initial and boundary conditions. Consequently a lot of attention has been paid to differential operators and they are the subject of much current research.

Most of the published work deals with the main problem just mentioned—the spectral theory of differential operators, i.e., the investigation of the spectra of differential operators and the expansion of given functions in terms of the eigenfunctions of these operators. Interest in such problems has been particularly stimulated by the development of quantum mechanics. The spectral theory of differential operators appears as the basic mathematical method for investigating many topics in quantum mechanics. In particular, quantum mechanics demands a detailed study of "singular" differential operators, of operators, for example, which are defined on some unbounded interval. Such operators may, in general, have a continuous spectrum as well as a discrete spectrum; and then an expansion in terms of eigenfunctions takes the form of a Stieltjes integral.

It should be mentioned that, in recent years, highly important results in this field have been obtained by Soviet mathematicians. Results such as the proof of the existence of an expansion in eigenfunctions of a singular, self-adjoint operator of arbitrary even order, the solution for the inversion of the Sturm-Liouville problem (see Chapter VIII), the completeness of the system of eigenfunctions and associated functions for a wide class of differential operators which are *not* self-adjoint, and many other results, are due, either entirely, or almost entirely, to the efforts of Soviet mathematicians.

Notwithstanding these fundamental achievements, the problems of spectral representation of differential operators cannot yet be regarded as exhausted. Particular attention should be drawn to the problem of determining the multiplicity of the spectrum and the deficiency indices of a differential operator in relation to the behaviour of its coefficients, and to the problem of selecting and normalizing a minimal system of eigenfunctions when the spectrum is continuous. In particular, very little work has been done for the case of an operator generated by a *system* of differential expressions, that is, for a differential operator in the space of vector-functions (cf. Chapters V and VII).

An equally important and interesting problem is that of expanding a given function in terms of eigenfunctions of a differential operator which is not self-adjoint. A noteworthy advance with this problem was made in a paper by M. V. Keldysh [47] for the case of a regular differential expression with, in general, non-regular boundary-conditions, but so far no one has investigated the case of a singular differential operator which is not self-adjoint.

Among other outstanding problems in the theory of differential operators we may also mention that of putting on a proper mathematical basis the so-called "perturbation" theory, which has many applications in quantum mechanics (see, in this connection, [39e], [56], [61], [96a], [112b], and [27b]).

Despite the importance of the theory of linear differential operators, due to its applications, and despite the abundance of fundamental results already achieved, the literature, both Soviet and foreign, is extremely poor in books devoted to this theory. Thus, in the foreign literature there is just one book in which a complete and exact account of the fundamental topics in the theory of second-order differential operators is given; this is [112a] by Titchmarsh, published in 1946. In the Soviet literature there is one monograph by B. M. Levitan [63a], published in 1950, in which another, and in fact a very elementary, presentation of the theory of differential operators is given. No book is yet available dealing with differential operators of higher order. Moreover, a whole range of questions arising in the theory of second-order differential operators are not discussed at all in the books by Titchmarsh and Levitan.

In the present book, the theory of linear, ordinary, differential operators of arbitrary order is developed. Extensive use is made of the ideas and theorems of functional analysis, especially those of the theory of linear operators in Hilbert space. For the reader's convenience all the necessary theorems in functional analysis are developed in the text itself, to make the book, as the author hopes, self-contained and accessible to a wide range of readers. Many topics in the theory of linear differential operators can also be discussed without using functional analysis, as Titchmarsh's treatment [112a] shows. However, the author does not consider such a treatment to be appropriate, since it is only by using the ideas and methods of functional analysis that a deeper understanding of the theory, and its most general results, can be achieved.

The book is in two parts. In the first part, which may be described as an elementary theory of differential operators, the use of the methods of functional analysis is kept to a minimum. This part presents the theory of differential operators defined on a finite interval, including the case of differential operators which are not self-adjoint, the theory being developed on the assumption that the coefficients of the operators are sufficiently smooth, i.e., sufficiently differentiable. To understand Part I the reader needs only an elementary knowledge of the theory of ordinary differential and integral equations and of the theory of functions.

Part II develops the theory of differential operators, using Hilbert space methods. Here, in addition to the subjects already mentioned, the reader will be expected to have some knowledge of the basic results in the theory of the Lebesgue integral.

Many new results are given in this book, most of them due to the work of Soviet mathematicians; in particular, the results relating to the deficiency indices and the spectrum of a differential operator.

The author wishes to express his thanks to M. I. Vishik for reading the book in manuscript and making a number of valuable suggestions for improving the text.

M. A. NAIMARK.

July, 1952.

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CHAPTER I

FUNDAMENTAL CONCEPTS AND THEOREMS

§1. *Definition and Principal Properties of the Linear Differential Operator*

1. General Definition of a Linear Vector-space and of a Linear Operator

A set R of elements $x,y,\ldots$ is called a *linear vector-space* if it is constituted as follows:

1°. *For any two elements $x,y \in R$ there is a sum $x + y$ having the following properties:*

- (a_1) if $x,y \in R$, then $x + y \in R$;
- (b_1) $x + y = y + x$;
- (c_1) $(x + y) + z = x + (y + z)$;
- (d_1) there is a "null element" 0 in R such that $x + 0 = x$ for all $x \in R$.

2°. *For each element $x \in R$ and for each real or complex number λ there is a product λx having the following properties:*

- (a_2) if $x \in R$, then $\lambda x \in R$;
- (b_2) $\lambda(\mu x) = (\lambda\mu)x$;
- (c_2) $1.x = x$;
- (d_2) $0.x = 0$ (where, on the L.H.S., 0 means the number 0, and on the R.H.S., 0 means the null-element);
- (e_2) $\lambda(x + y) = \lambda x + \lambda y$;
- (f_2) $(\lambda + \mu)x = \lambda x + \mu x$.

The element $(-1)x$ is denoted by $-x$, and by properties (c_2), (f_2) and (d_2)

$$x + (-x) = (1 + (-1))x = 0.x = 0.$$

The elements $x,y,\ldots$ of the space R are called *vectors* in R.

In this definition the nature of the elements $x,y,\ldots$ is entirely arbitrary, and it is equally a matter of indifference how the concepts of the sum $x + y$ and the product of an element x with a number λ are interpreted in particular cases. It is required only that these concepts shall satisfy the conditions enumerated above. Any two operations whatever which satisfy these conditions may be regarded respectively as addition of vectors, and as multiplication of a vector by a number; and the set of elements for which these operations are defined may then be called a linear vector space.

If in the space R multiplication only by real numbers is allowed, then R is called a real space; but if multiplication by any complex number is allowed, then R is called a complex space. We shall take R to be a complex space unless the contrary is specifically stated. An expression of the form $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n$ is called a *linear combination* of the vectors $x_1, x_2, \ldots, x_n$; a linear combination is said to be *trivial* if all the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ are zero, otherwise it is *non-trivial.* The vectors $x_1, x_2, \ldots, x_n$ are said to be *linearly dependent* if at least one non-trivial combination of them vanishes, and to be *linearly independent* if no non-trivial combination of them vanishes.

A space R is *finite-dimensional*, or more precisely, *n-dimensional* if in it there are n but not more than n linearly independent vectors. Such a system of n vectors is called a *basis* in R. If, on the other hand, there are arbitrarily many linearly independent vectors in R, then R is *infinite-dimensional.*

A subset R' of the finite-dimensional or infinite-dimensional space R is called a *subspace*[1] of R if every linear combination of elements of R' is itself an element of R'.

Let $\mathscr{D}$ be a subset in the linear space R. A function A which assigns to each element x of $\mathscr{D}$ some element, $x' = A(x)$, of R is called an *operator* in the space R and $\mathscr{D}$ is the *domain of definition* of A. Often we write Ax instead of $A(x)$ if this is not likely to be misunderstood; and we write $\mathscr{D}_A$ instead of $\mathscr{D}$ when it is necessary to emphasize that $\mathscr{D}$ is precisely the domain of definition of the operator A.

The set of all vectors $Ax, x \in \mathscr{D}_A$, is called the *range of values* of the operator A and is denoted by $\mathscr{R}_A$ or $A\mathscr{D}_A$.

If, generally, $\mathscr{E}$ is an arbitrary set, then $A\mathscr{E}$ denotes the set of all vectors $Ax, x \in \mathscr{E}$, i.e. the set of all vectors which are obtained by applying the operator A to all the vectors of the set $\mathscr{E}$.

An operator A is said to be *linear* if $\mathscr{D}_A$ is a subspace and if, for arbitrary vectors $x, y \in \mathscr{D}_A$ and for any number λ, the relations

$$A(\lambda x) = \lambda A(x),$$

$$A(x + y) = A(x) + A(y)$$

hold.

Two operators A and B in the space R are regarded as equal if and only if they are defined on the same domain $\mathscr{D}$ and if $Ax = Bx$ for all $x \in \mathscr{D}$.

An operator A is called an *extension* of an operator B and we write $A \supset B$ or $B \subset A$ if $\mathscr{D}_A \supset \mathscr{D}_B$ and if the operators are equal in $\mathscr{D}_B$, i.e. if $Ax = Bx$ for all $x \in \mathscr{D}_B$. In such a case we also call the operator B a *restriction* of the operator A to $\mathscr{D}_B$.

In this book we shall consider only operators which are linear, and therefore, to avoid constant repetition of the word "linear", we shall always use the one word "operator", but mean by it "linear operator".

---

[1] In the English literature this is usually called a *linear manifold.*

2. Linear Differential Expressions

A *linear differential expression* means an expression of the form

$$l(y) = p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y. \tag{1}$$

The functions $p_0(x),p_1(x),\dots,p_n(x)$ are called the *coefficients* and the number n is called the *order* of the differential expression.

In this first chapter we shall assume that the functions $\frac{1}{p_0(x)},p_1(x),p_2(x),\dots$ $p_n(x)$ are continuous on a fixed, closed, finite interval $[a,b]$; in some cases we shall impose further conditions.

We denote by $C^{(n)}$ the set of all functions $y(x)$ which have continuous derivatives up to the nth order inclusive on the interval $[a,b]$.

For every function $y \in C^{(n)}$ the differential expression $l(y)$ is well defined, and represents a function continuous on the interval $[a,b]$.

3. Boundary Conditions

We denote the values of the function y and its first $(n - 1)$ successive derivatives at the boundary points a and b of the interval $[a,b]$ by

$$y_a,y'_a,\dots,y_a^{(n-1)};\ y_b,y'_b,\dots,y_b^{(n-1)}. \tag{2}$$

Let $U(y)$ be a linear form in the variables (2):

$$U(y) = \alpha_0 y_a + \alpha_1 y'_a + \dots + \alpha_{n-1}y_a^{(n-1)} + \beta_0 y_b + \beta_1 y'_b + \dots + \beta_{n-1}y_b^{(n-1)}. \tag{3}$$

If several such forms $U_\nu(y)$ have been specified, $\nu = 1,\dots,m$, and if the conditions

$$U_\nu(y) = 0, \qquad \nu = 1,\dots,m, \tag{4}$$

are imposed on the functions $y(x) \in C^{(n)}$, we call these the *boundary conditions* which the functions y must satisfy.

We denote by $\mathscr{D}$ the set of all functions $y \in C^{(n)}$ which satisfy a specified system of boundary conditions of the form (4). Clearly $\mathscr{D}$ is a linear subspace in $C^{(n)}$ which coincides with $C^{(n)}$ only if the conditions (4) are entirely lacking or if all their coefficients vanish.

Suppose a certain differential expression $l(y)$ and a particular subspace $\mathscr{D}$ defined by conditions of the form (4) are given. To each function $y \in \mathscr{D}$ we let the function $u = l(y)$ correspond. This relation is a linear operator with $\mathscr{D}$ as its domain of definition; we denote it by L. Using the notation previously introduced, we write

$$u = Ly.$$

The operator L is called *the differential operator generated by the differential expression $l(y)$ and the boundary conditions (4).*

In this way—and this fact will be very important in the sequel—*one and the same differential expression can generate various differential operators depending on the way in which the boundary conditions (4) are chosen.*

If, in particular, the conditions (4) are absent, we obtain a differential operator, denoted by L_1, with the domain of definition $\mathscr{D} = C^{(n)}$. L_1 is obviously an extension of all other operators L which can be generated by the same differential expression $l(y)$.

For many questions it turns out to be profitable to investigate not merely the operator L_1 which has the widest domain of definition, but also the operators considered above, which are restrictions of L_1.

Certain of the forms $U_\nu(y)$ may be expressible as linear combinations of the remaining forms. The corresponding conditions $U_\nu(y) = 0$ are then immediate consequences of the remaining conditions, and therefore, being superfluous, can be discarded. Hence we may, *ab initio*, regard the forms $U_\nu(y)$ as being linearly independent. This implies, of course, that the rank of the matrix formed from the coefficients of these forms is equal to m.

For $m = 2n$, the equations (4) are equivalent to

$$y_a = y_a' = \ldots = y_a^{(n-1)} = y_b = y_b' = \ldots = y_b^{(n-1)} = 0.$$

The particular differential operator generated by $l(y)$ and these conditions is denoted by L_0.

We can also give the following geometrical interpretation of the boundary conditions. Let η stand for the whole of system (2): $\eta = (y_a, y_a', \ldots, y_a^{(n-1)}, y_b, y_b', \ldots, y_b^{(n-1)})$. Clearly we may regard η as a vector in the $2n$-dimensional space R_{2n}. The conditions (4) separate out from the space R_{2n} a definite subspace $\mathscr{M}$ of dimensions $(2n - m)$. These conditions imply that the vector η which corresponds to a function y belongs to the subspace $\mathscr{M}$.

4. The Homogeneous Boundary-Value Problem

The problem of determining a function $y \in C^{(n)}$ which satisfies the conditions

$$l(y) = 0, \tag{5}$$

$$U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,m. \tag{6}$$

is called the *homogeneous boundary-value problem*. If L is the operator which is generated by the differential expression $l(y)$ and the boundary conditions (6), then the homogeneous boundary-value problem amounts to finding, in the domain of definition $\mathscr{D}$ of the operator L, a function y for which L vanishes.

Clearly, any homogeneous boundary-value problem always has at least one solution, viz. the solution $y = 0$. This is called the *trivial* solution. A homogeneous boundary-value problem can, however, have other "non-trivial" solutions, i.e. solutions which are not identically zero.

We now want to ascertain under what conditions a homogeneous boundary-value problem has non-trivial solutions.

Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the differential equation $l(y) = 0$. Then, as is known from the theory of linear differential equations (see [108] or [17], for instance), *any* solution of the equation $l(y) = 0$, and therefore the solution of the homogeneous boundary-value problem, too, must be expressible in the form

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n,$$

where $c_1, c_2, \ldots, c_n$ are certain constants. By substitution in the conditions (6) we obtain a system of linear, homogeneous equations

$$\left.\begin{array}{l} c_1 U_1(y_1) + c_2 U_1(y_2) + \ldots + c_n U_1(y_n) = 0 \\ c_1 U_2(y_1) + c_2 U_2(y_2) + \ldots + c_n U_2(y_n) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \qquad \cdot \\ c_1 U_m(y_1) + c_2 U_m(y_2) + \ldots + c_n U_m(y_n) = 0 \end{array}\right\} \tag{7}$$

for the determination of the constants $c_1, c_2, \ldots, c_n$. Let r be the rank of the matrix

$$U = \begin{bmatrix} U_1(y_1) & U_1(y_2) & & U_1(y_n) \\ U_2(y_1) & U_2(y_2) & & U_2(y_n) \\ \cdot & \cdot & & \\ U_m(y_1) & U_m(y_2) & & U_m(y_n) \end{bmatrix} \tag{8}$$

Then there are precisely $(n - r)$ independent solutions of the equations (7) for the $c_1, c_2, \ldots, c_n$; and to these correspond $(n - r)$ independent solutions y of the boundary-value problem.

Hence the following result:

I. *If the rank of the matrix U is equal to r, the homogeneous, boundary-value problem has exactly $(n - r)$ independent solutions.*

In particular,

II (a). *The homogeneous, boundary-value problem has a non-trivial solution if and only if the rank r of the matrix U is less than the order n of the differential expression l.*

(b). *For $m < n$, the homogeneous, boundary-value problem always has a non-trivial solution.*

(c). *For $m = n$, the homogeneous, boundary-value problem has a non-trivial solution if and only if the determinant of the matrix U (in this case, a square matrix) vanishes.*

The rank of the matrix U does not depend on the choice of the basic system

$y_1,\ldots,y_n$. For, the transition from one such system $y_1,\ldots,y_n$ to another $\hat{y}_1,\ldots,\hat{y}_n$ is effected by means of a linear transformation

$$\hat{y}_i = \sum_{j=1}^{n} a_{ij} y_j, \qquad i = 1,2,\ldots,n,$$

with a non-zero determinant. Since in this transition the matrix U is multiplied by the matrix $[a_{ij}]$ $(i,j = 1,\ldots,n)$, the rank of U remains unchanged.

The rank of the matrix U is called *the rank of the boundary-value problem.*

The definition which we have given of the homogeneous boundary-value problem may be generalized as follows.

Let $U_1(y), U_2(y),\ldots,U_n(y)$ be linearly independent functionals in $C^{(n)}$. (As the norm in $C^{(n)}$, we can take $|y| = \sum_{k=0}^{n} \max_{a \le x \le b} |y^{(k)}(x)|$.) Let $l(y)$ be a differential expression of the nth order.

Then by a generalized homogeneous boundary-value problem we mean the problem of determining a function $y \in C^{(n)}$ which satisfies the conditions

$$l(y) = 0, \qquad U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n.$$

The results of this section will clearly go over completely to the case of a generalized boundary-value problem.

Certain special cases of the generalized boundary-value problem are treated in Tamarkin's articles [110] and by other writers (see [106] and [45a], Part II, §5 in which a detailed list of references is given); but the question has not so far been discussed in the full generality given here.

5. Lagrange's Formula. The Adjoint Differential Expression

We now assume that the coefficients $p_k(x), k = 0,1,2,\ldots,n$, of the differential expression

$$l(y) = p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + p_n(x)y$$

have continuous derivatives up to the order $(n - k)$ inclusive on the interval $[a,b]$. Further let y and z be two arbitrary functions in $C^{(n)}$. By k partial integrations we get

$$\int_a^b p_{n-k}\bar{z}y^{(k)}dx = \Big[p_{n-k}\bar{z}y^{(k-1)} - (p_{n-k}\bar{z})'y^{(k-2)} + \ldots \\ + (-1)^{k-1}(p_{n-k}\bar{z})^{(k-1)}y\Big]_{x=a}^{x=b} + (-1)^k \int_a^b y(p_{n-k}\bar{z})^{(k)}dx. \quad (9)$$

(Here $\bar{z}$ denotes the complex number conjugate to z, and $\bar{z} = \overline{z(x)}$ denotes the function whose values are conjugate to those of $z(x)$.)

If we put $k = n, n-1, \ldots, 0$ in (9) and add the resulting equations, we obtain the formula

$$\int_a^b l(y)\bar{z}dx = P(\eta,\zeta) + \int_a^b y\overline{l^*(z)}dz, \tag{10}$$

where

$$l^*(z) = (-1)^n(\bar{p}_0 z)^{(n)} + (-1)^{n-1}(\bar{p}_1 z)^{(n-1)} + (-1)^{n-2}(\bar{p}_2 z)^{(n-2)} + \ldots + \bar{p}_n z, \tag{11}$$

and $P(\eta,\zeta)$ is a certain bilinear form in the variables

$$\eta = (y_a, y'_a, \ldots, y_a^{(n-1)}, y_b, y'_b, \ldots, y_b^{(n-1)})$$

and

$$\zeta = (z_a, z'_a, \ldots, z_a^{(n-1)}, z_b, z'_b, \ldots, z_b^{(n-1)}).$$

The differential expression $l^*(z)$ defined by the formula (11) is called the *adjoint* differential expression of $l(y)$, and (10) is called *Lagrange's formula.*

If we apply the preceding considerations to the integral

$$\int_a^b l^*(z).\bar{y}dx,$$

we obtain an expression of the form

$$\int_a^b l^*(z)\bar{y}dx = Q(\eta,\zeta) + \int_a^b z.\overline{l(y)}dx.$$

Hence the differential expression $l(y)$ is adjoint to $l^*(z)$:

$$l^{**}(y) = l(y).$$

In other words, *the differential expressions* $l(y)$ *and* $l^*(y)$ *are mutually adjoint.*

From the definition (11) of adjoint expressions we see at once that

$$(l_1 + l_2)^* = l_1^* + l_2^*, \tag{12}$$

and, if λ is any number,

$$(\lambda l)^* = \bar{\lambda} l^*. \tag{13}$$

A differential expression $l(y)$ is said to be *self-adjoint*[1] if $l^* = l$. It follows from (12) and (13) that:

III (a). *A sum of self-adjoint differential expressions is also a self-adjoint differential expression.*

(b). *The product of a self-adjoint differential expression with a real number is also a self-adjoint differential expression.*

We now want to determine the general form of all self-adjoint differential expressions.

IV. *Any self-adjoint differential expression is a sum of differential expressions of the form*

$$l_{2\nu}(y) = (py^{(\nu)})^{(\nu)};\ l_{2\nu-1}(y) = \tfrac{1}{2}[(ipy^{(\nu-1)})^{(\nu)} + (ipy^{(\nu)})^{(\nu-1)}],$$

where p *in each case is a real-valued function on* $[a,b]$.

---

[1] In the English literature such an expression is often said to be *formally self-adjoint.*

Proof. We consider the integrals

$$\int_a^b l_{2\nu}(y)\bar{z}dx \text{ and } \int_a^b l_{2\nu-1}(y)\bar{z}dx, \text{ with } y,z \in C^{(n)}.$$

By partial integration it is at once clear that $l_{2\nu}(y)$ and $l_{2\nu-1}(y)$ are self-adjoint differential expressions. Consequently a sum of expressions of the form $l_{2\nu}(y)$ and $l_{2\nu-1}(y)$ is likewise a self-adjoint differential expression.

Conversely, let

$$l(y) = p_0 y^{(n)} + p_1 y^{(n-1)} + \dots + p_n y$$

be a self-adjoint differential expression; we shall prove that $l(y)$ can be represented as the sum of expressions of the form $l_{2\nu}(y)$ and $l_{2\nu-1}(y)$.

By definition, $l(y)$ must agree with the adjoint expression

$$\begin{aligned} l^*(y) &= (-1)^n(\bar{p}_0 y)^{(n)} + (-1)^{n-1}(\bar{p}_1 y)^{(n-1)} + \dots + \bar{p}_n y \\ &= (-1)^n \bar{p}_0 y^{(n)} + [(-1)^{n-1} n \bar{p}_0' + (-1)^{n-1}\bar{p}_1] y^{(n-1)} + \dots \end{aligned}$$

Hence, in particular,

$$p_0 = (-1)^n \bar{p}_0. \tag{14}$$

If n is even and $= 2\mu$, (14) implies $p_0 = \bar{p}_0$, and so p_0 takes only real values. If we subtract from $l(y)$ the self-adjoint expression

$$l_{2\mu}(y) = (p_0 y^{(\mu)})^{(\mu)} = p_0 y^{(n)} + (\mu - 1)p_0' y^{(n-1)} + \dots,$$

we obtain the self-adjoint expression $l(y) - l_{2\mu}(y)$ of order $n - 1$.

Next, if n is odd and $= 2\mu - 1$, (14) implies $p_0 = -\bar{p}_0$, and therefore $p_0 = ip$, where p takes only real values. By subtracting from $l(y)$ the self-adjoint expression

$$\begin{aligned} l_{2\mu-1}(y) &= \tfrac{1}{2}[(ipy^{(\mu-1)})^{(\mu)} + (ip^{(\mu)})^{(\mu-1)}] \\ &= (ip)y^{(2\mu-1)} + \tfrac{1}{2}\mu ip' y^{(2\mu-2)} + \dots \\ &= p_0 y^{(n)} + \tfrac{1}{2}\mu p_0' y^{(n-1)} + \dots, \end{aligned}$$

we obtain the self-adjoint expression $l(y) - l_{2\mu-1}(y)$ of order $(n - 1)$.

It follows from this argument that if self-adjoint expressions of the form $l_{2\nu}(y)$ and $l_{2\mu-1}(y)$ are subtracted successively from $l(y)$, then finally a self-adjoint expression of zero order will be obtained. This must necessarily agree with $l_0(y) = py$, where p denotes a real function. This proves assertion IV.

If, in particular, $l(y)$ is a self-adjoint expression with real coefficients, then it can have no addends of the form $l_{2\mu-1}(y)$. Hence

V. *Any self-adjoint differential expression with real coefficients is necessarily of even order and has the form*

$$l(y) = (p_0 y^{(\mu)})^{(\mu)} + (p_1 y^{(\mu-1)})^{(\mu-1)} + \dots + (p_{\mu-1} y')' + p_\mu y$$

where $p_0, p_1, \dots, p_\mu$ *are real-valued functions.*

6. Adjoint Boundary Conditions. The Adjoint Operator

Let $U_1,\dots,U_m$ be linearly independent forms in the variables $y_a,y_a',\dots,y_a^{(n-1)}$, $y_b,y_b',\dots,y_b^{(n-1)}$; if $m < 2n$, we shall supplement them with other forms $U_{m+1},\dots,U_{2n}$ to obtain a linearly independent system of $2n$ forms $U_1,U_2,\dots,U_{2n}$. Since these forms are linearly independent, the variables $y_a,y_a',\dots,y_a^{(n-1)}$, $y_b,y_b',\dots,y_b^{(n-1)}$ can be expressed as linear combinations of the forms $U_1,U_2,\dots,U_{2n}$.

We substitute these expressions in the bilinear form $P(\eta,\zeta)$ which occurred in Lagrange's formula (see formula (10)). Then $P(\eta,\zeta)$ becomes a linear, homogeneous form in the variables $U_1,U_2,\dots,U_{2n}$, and its $2n$ coefficients are themselves linear, homogeneous forms, which we denote by $V_{2n},V_{2n-1},\dots,V_1$, in the variables $z_a,z_a',\dots,z_a^{(n-1)},z_b,z_b',\dots,z_b^{(n-1)}$. Lagrange's formula now takes the form

$$\int_a^b l(y).\bar{z}dx = U_1V_{2n} + U_2V_{2n-1} + \dots + U_{2n}V_1 + \int_a^b y\overline{l^*(z)}dx \qquad (15)$$

The forms $V_1,V_2,\dots,V_{2n}$ *are linearly independent.*

To prove this assertion, we first recall that the form $P(\eta,\zeta)$ arose from the substitution of the limits $x = a$ and $x = b$ in the integrated terms obtained in the integration by parts, and so it has the form

$$P(\eta,\zeta) = P_b(\eta,\zeta) - P_a(\eta,\zeta),$$

and $P_a(\eta,\zeta)$ and $P_b(\eta,\zeta)$ involve only the values of the functions of y,y', $\dots,y^{(n-1)},z,z',\dots,z^{(n-1)}$ at the points $x = a$ and $x = b$ respectively.

Consequently the matrix of $P(\eta,\zeta)$ is of the form

$$\begin{bmatrix} -\Delta_a & 0 \\ 0 & \Delta_b \end{bmatrix}, \qquad (16)$$

where Δ_a and Δ_b denote the matrices of $P_a(\eta,\zeta)$ and $P_b(\eta,\zeta)$ respectively, and 0 the null matrix.

But from formula (9) we see at once that each of the matrices Δ_a and Δ_b has the form

$$\begin{bmatrix} . & . & \dots & (-1)^{n-1}p_0 \\ . & . & ..(-1)^{n-2}p_0 & 0 \\ . & . & \dots & . \\ . & . & \dots & . \\ . & -p_0 & ..0 & 0 \\ p_0 & 0 & ..0 & 0 \end{bmatrix},$$

where p_0 is to be evaluated at $x = a$ and $x = b$ for Δ_a and Δ_b respectively. (The elements above the diagonal in this matrix have not been written down, since they play no part in the argument.)

The determinants of both these matrices are non-zero, and from (16) it follows that *the determinant of* $P(\eta,\zeta)$ *is also non-zero.*

The transition from the variables $y_a, y'_a, \ldots, y_a^{(n-1)}, y_b, y'_b, \ldots, y_b^{(n-1)}$ to the variables $U_1, U_2, \ldots, U_{2n}$ is a linear transformation with non-zero determinant. Hence the determinant of the matrix of P in the variables

$$U_1, U_2, \ldots, U_{2n} \text{ and } z_a, z'_a, \ldots, z_a^{(n-1)}, z_b, z'_b, \ldots, z_b^{(n-1)}$$

is also non-zero. But this matrix is also the coefficient matrix of the forms

$$V_{2n}, V_{2n-1}, \ldots, V_1.$$

These forms are therefore linearly independent.

The boundary conditions

$$V_1 = 0, V_2 = 0, \ldots, V_{2n-m} = 0 \tag{17}$$

(and all boundary conditions equivalent to them) are said to be *adjoint* to the original boundary conditions.

$$U_1 = 0, U_2 = 0, \ldots, U_m = 0. \tag{18}$$

Boundary conditions are *self-adjoint* if they are equivalent to their adjoint boundary conditions.

Let L be the operator generated by the expression $l(y)$ and the boundary conditions (18). The operator generated by $l^*(y)$ and the boundary conditions (17) will be denoted by L^* and called the *adjoint operator* to L.

It follows from formula (15) and the boundary conditions (17) and (18) that *the equation*

$$\int_a^b Ly . \bar{z} dx = \int_a^b y . \overline{L^* z} dx \tag{19}$$

holds for the operators L *and* L^*, for all y in the domain of definition of L and all z in the domain of definition of L^*.

If we introduce the contracted notation

$$(y,z) = \int_a^b y(x) . \overline{z(x)} dx,$$

then (19) takes the form

$$(Ly,z) = (y,L^*z). \tag{20}$$

It follows from the definition of adjoint operators that the operator L is adjoint to the operator L^*:

$$L^{**} = L.$$

An operator is *self-adjoint* if $L^* = L$.

It also follows from the definition of L^* that:

An operator L is self-adjoint if and only if it is generated by a self-adjoint differential expression and self-adjoint boundary conditions.

For a self-adjoint operator L the formula (20) becomes

$$(Ly,z) = (y,Lz). \tag{21}$$

7. The Adjoint Boundary-Value Problem

If L^* is the operator adjoint to L, the homogeneous boundary-value problem

$$L^*z = 0 \tag{22}$$

is said to be *adjoint* to the boundary-value problem

$$Ly = 0. \tag{23}$$

In more detail, the boundary-value problem (22) can be written as:

$$l^*(z) = 0, \tag{24}$$

$$V_\nu(z) = 0, \qquad \nu = 1,2,\dots,2n - m. \tag{25}$$

where $l^*(z)$ is the differential expression adjoint to $l(y)$, and (25) are the boundary conditions adjoint to those of L.

We intend now to obtain a relation between the ranks of adjoint boundary-value problems.

Let $z_1,z_2,\dots,z_n$ be linearly independent solutions of the equation $l^*(z) = 0$, and let r' be the rank of the matrix

$$\begin{bmatrix} V_1(z_1) & & V_1(z_n) \\ V_2(z_1) & \dots & V_2(z_n) \\ \cdot & \dots & \cdot \\ V_{2n-m}(z_1) & \dots & V_{2n-m}(z_n) \end{bmatrix} \tag{26}$$

Then the adjoint boundary-value problem (24),(25) has exactly $(n - r')$ independent solutions.

If, on the other hand, y is an arbitrary solution of the boundary-value problem (23), then the left-hand side of Lagrange's formula (15) vanishes for $z = z_\nu$. Also, $U_1(y) = \dots = U_m(y) = 0$. Hence in this case Lagrange's formula simplifies to

$$U_{m+1}(y)V_{2n-m}(z_\nu) + \dots + U_{2n}(y)V_1(z_\nu) = 0.$$

Putting $\nu = 1,2,\dots,n$, we obtain for the forms $U_{m+1}(y),\dots,U_{2n}(y)$ the system of equations

$$\left.\begin{aligned} U_{m+1}(y)V_{2n-m}(z_1) + \dots + U_{2n}(y)V_1(z_1) &= 0, \\ U_{m+1}(y)V_{2n-m}(z_2) + \dots + U_{2n}(y)V_1(z_2) &= 0, \\ \cdot \qquad\qquad\qquad \cdot \qquad & \quad \cdot \\ U_{m+1}(y)V_{2n-m}(z_n) + \dots + U_{2n}(y)V_1(z_n) &= 0. \end{aligned}\right\} \tag{27}$$

This system has at least $(n - r)$ independent solutions, viz.

$$U_{m+1}(y_j),\ldots,U_{2n}(y_j), \qquad j = 1,2,\ldots,n - r,$$

if r is the rank of the boundary-value problem (23), and $y_1,\ldots,y_{n-r}$ is a system of linearly independent solutions of (23). Therefore the rank of its matrix is not greater than $2n - m - (n - r) = n - m + r$. Since the matrix of the system (27) corresponds—up to the arrangement of rows and columns—to the matrix (26), we have

$$r' \leqslant n - m + r. \tag{28}$$

We now recall that the two boundary-value problems are mutually adjoint, and therefore we can interchange their roles. In such an interchange, r and r' are interchanged, and m goes over into $(2n - m)$; hence $r \leqslant n - (2n - m) + r'$, i.e.

$$r \leqslant m - n + r'. \tag{29}$$

Comparing (28) and (29), we have

$$r' = n - m + r. \tag{30}$$

VI. *The rank r of a boundary-value problem is related to the rank r' of the adjoint boundary-value problem by the relation*

$$r' = n - m + r.$$

For the particular case $m = n$, formula (30) gives $r' = r$. Hence:

VII. *If the number of independent boundary conditions is equal to the order of the differential expression, then the rank of the boundary-value problem is equal to the rank of the adjoint boundary-value problem.*

In particular it follows from this that:

VIII. *If the number of boundary conditions is equal to the order of the differential expression, and if the associated homogeneous boundary-value problem has only the trivial solution, then the adjoint boundary-value problem has only the trivial solution.*

Additional information on the topics discussed in this section can be found in chapter 11 of Coddington and Levinson's book [17].

We have been considering here problems where the boundary conditions are prescribed at the end-points of an interval. In mathematical physics and in applications, however, problems are encountered where the boundary conditions are prescribed at n arbitrary points of an interval $[a,b]$. Chapter 4 of Sansone's book [97] may be consulted on these so-called n-point problems.

§2. *Eigenvalues and Eigenfunctions of a Differential Operator*

1. Definition of Eigenvalues and Eigenfunctions

A number λ is called an *eigenvalue* of an operator L if there exists in the domain of definition of the operator L a function $y \not\equiv 0$ such that

$$Ly = \lambda y. \tag{1}$$

The function y is called the *eigenfunction*, of the operator L, for the eigehvalue λ.

The operator L may be generated by the differential expression $l(y)$ and the boundary-conditions

$$U_1(y) = 0,\ldots,U_n(y) = 0. \tag{2}$$

Since an eigenfunction y must belong to the domain of definition of the operator L, it must satisfy the conditions (2). Moreover, $Ly = l(y)$, and therefore (1) is equivalent to

$$l(y) = \lambda y. \tag{3}$$

Hence:

The eigenvalues of an operator L are those values of the parameter λ for which the homogeneous boundary-value problem

$$l(y) = \lambda y, \qquad U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,m, \tag{4}$$

has non-trivial solutions; each of these non-trivial solutions is an eigenfunction belonging to λ.

A linear combination of eigenfunctions which belong to one and the same eigenvalue λ is itself an eigenfunction belonging to λ. For, if $Ly_1 = \lambda y_1$ and $Ly_2 = \lambda y_2$, then $L(c_1y_1 + c_2y_2) = \lambda(c_1y_1 + c_2y_2)$ for any constants c_1 and c_2.

Since a homogeneous equation (3) can have, for a given λ, not more than n linearly independent solutions, it follows that *the aggregate of all eigenfunctions which belong to one and the same eigenvalue form a finite-dimensional space of dimension $\leqslant n$.* The dimension of this space is simply the number of linearly independent solutions of the boundary-value problem (4) for the given eigenvalue λ; this number is called the *multiplicity* of the eigenvalue.

We shall specify conditions for the determination of eigenvalues.

Let

$$y_1(x,\lambda),y_2(x,\lambda),\ldots,y_n(x,\lambda) \tag{5}$$

denote a fundamental system of solutions of the differential equation (3) which satisfy the initial conditions

$$y_j^{(\nu-1)}(a,\lambda) = \begin{cases} 0 \text{ for } j \neq \nu, \\ 1 \text{ for } j = \nu, \end{cases} \qquad j,\nu = 1,2,\ldots,n. \tag{6}$$

From general theorems on the solutions of linear differential equations it follows that, for each fixed value of x in $[a,b]$, the functions (5) are integral, analytic functions of the parameter λ.

From the results of §1.4, the boundary-value problem (4) has a non-trivial solution if and only if the rank r of the matrix

$$\mathcal{U} = \begin{bmatrix} U_1(y_1) & \cdots & U_1(y_n) \\ \cdot & & \\ U_m(y_1) & \cdots & U_m(y_n) \end{bmatrix}$$

is less than n:

and if $m < n$, then $r < n$; and in this case the boundary-value problem (4) has a non-trivial solution for any value of λ. Hence, *if* $m < n$, *any value of* λ *is an eigenvalue.*

If $m \geqslant n$, the rank of the matrix $\mathcal{U}$ will be less than n if and only if all its minors of order n vanish. But each of these minors is an integral, analytic function of λ. Hence the only possible alternatives are:

1°. *All the nth-order minors of the matrix* $\mathcal{U}$ *vanish identically, and so, by the previous result, any value of* λ *is an eigenvalue.*

2°. *At least one nth-order minor of* $\mathcal{U}$ *does not vanish identically.* In this case, only the zeros of these minors can be eigenvalues, and further, a zero of one particular minor can be an eigenvalue only if it makes all the other, not identically zero, minors of order n of $\mathcal{U}$ also vanish.

Now, a non-vanishing integral function has at most denumerably many zeros (it need not have any at all), and moreover, these zeros have no finite point of accumulation. Hence: *In case* 2°, *the operator L has at most denumerably many eigenvalues* (*it may have none at all*), *and these eigenvalues have no finite limit-point.*

Summarizing, then, we have the following alternatives.

I. *For any differential operator L, only the following two possibilities can occur:*

   1°. *Every number* λ *is an eigenvalue for* L;

   2°. *The operator has at most denumerably many eigenvalues* (*in particular cases, none at all*), *and these eigenvalues can have no finite limit-point.*

The case when $m = n$ is of particular interest, and in the sequel we shall consider this particular case, if nothing to the contrary is stated.

We put, then,

$$\Delta(\lambda) = \begin{bmatrix} U_1(y_1) & & U_1(y_n) \\ \cdot & \cdots & \cdot \\ U_n(y_1) & & U_n(y_n) \end{bmatrix} \tag{7}$$

By the preceding discussion, $\Delta(\lambda)$ is an integral, analytic function of λ, the so-called *characteristic determinant* of the operator L (or of the boundary-value problem $Ly = 0$); and the following theorems hold:

II. *The eigenvalues of the operator L are the zeros of the function* $\Delta(\lambda)$. *If* $\Delta(\lambda)$ *vanishes identically, then any number* λ *is an eigenvalue of the operator L.*

If, however, $\Delta(\lambda)$ *is not identically zero, the operator L has at most denumerably many eigenvalues, and these eigenvalues can have no finite limit-point.*

If, in particular, the function $\Delta(\lambda)$ has no zeros at all, then the operator L has no eigenvalues.

An eigenvalue λ may be a multiple zero of $\Delta(\lambda)$.

III. *If* λ_0 *is a zero of the characteristic determinant* $\Delta(\lambda)$ *with multiplicity* ν, *then the multiplicity of the eigenvalue* λ_0 *cannot be greater than* ν.

Proof. Let r be the rank of the matrix belonging to the determinant $\Delta(\lambda_0)$; then the multiplicity of the eigenvalue λ_0 is equal to $(n - r)$. On the other hand, from a well-known rule for the differentiation of determinants, all the derivatives of the determinant $\Delta(\lambda)$ up to the $(n - r - 1)$th order inclusive vanish for $\lambda = \lambda_0$. Since λ_0 is a zero of multiplicity ν, it follows that $n - r - 1 \leqslant \nu - 1$, and hence $n - r \leqslant \nu$.

If, in particular, $\nu = 1$, then $n - r \leqslant 1$; on the other hand, $n - r \geqslant 1$, since $\Delta(\lambda_0) = 0$. Hence $n - r = 1$, and we have proved the following:

IV. *If* λ_0 *is a simple zero of the characteristic determinant* $\Delta(\lambda)$, *then the multiplicity of the eigenvalue* λ_0 *of the operator L is also unity.* An eigenvalue is called *simple* if it is a simple zero of the characteristic determinant $\Delta(\lambda)$.

2. Various Generalizations of Eigenvalue Problems

In applications, more general eigenvalue problems occur from time to time. The most general formulation of the problem is as follows:

The coefficients of a linear differential expression $l(y)$, *and perhaps also those of the boundary conditions* $U_\nu(y) = 0, \nu = 1,2,\dots,n$, *may depend on a certain complex parameter* λ. *The values of* λ *are to be determined for which the boundary-value problem*

$$l(y) = 0, \qquad U_\nu(y) = 0, \qquad \nu = 1,2,\dots,n \tag{8}$$

has a non-trivial solution.

These values of the parameter λ are called the *eigenvalues* of the boundary-value problem (8), and the corresponding non-trivial solutions y are called its *eigenfunctions*.

Using the same considerations as in the preceding subsection, we construct the characteristic determinant $\Delta(\lambda)$ according to formula (7); *the zeros of* $\Delta(\lambda)$ *will again be eigenvalues.*

If the coefficients of the expression $l(y)$ and of the boundary-conditions are analytic functions of the parameter λ which are regular in a certain domain $\mathscr{G}$ of the λ-plane, then the determinant $\Delta(\lambda)$ will also be a regular, analytic function of λ in the domain $\mathscr{G}$. There are then the two following possibilities:

1°. $\Delta(\lambda) \equiv 0$ *in the domain* $\mathscr{G}$; then every number $\lambda \in \mathscr{G}$ is an eigenvalue.

2°. $\Delta(\lambda) \not\equiv 0$; there then exist in the domain $\mathscr{G}$ at most denumerably many eigenvalues and these have no point of accumulation within $\mathscr{G}$.

If, in particular, the coefficients of the expression $l(y)$ and the forms $U_\nu(y)$ are integral, analytic functions of λ, then $\Delta(\lambda)$ is also an integral function; and hence in this case the theorems enunciated at the end of the last subsection remain valid. The most commonly occurring case is where $l(y) = l_1(y) + \lambda l_2(y)$, and both $l_1(y)$ and $l_2(y)$ and the forms $U_\nu(y)$ are independent of λ. The coefficients of the expression $l(y)$ are then linear (and therefore integral) functions of the parameter λ. So in this case, too, the theorems states in subsection 2 hold.

We may mention, further, the following important generalization of the eigenvalue problem defined above:

$$l_1(y) + \lambda l_2(y) = 0, \qquad U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n,$$

where l_1, l_2 are self-adjoint differential expressions and $U_\nu(y) = 0\,(\nu = 1,2,\ldots,n)$ are self-adjoint boundary-conditions relative to the differential expressions $l_1(y)$ and $l_2(y)$. This problem has been analyzed in detail by E. Kamke (see [45]).

3. Associated Functions

We now turn our attention to the generalized eigenvalue problem

$$l(y) = 0, \qquad U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n.$$

for which the coefficients of the differential expression $l(y)$ and of the forms $U_\nu(y)$ are integral analytic functions of the parameter λ. Let λ_0 be an eigenvalue and $\varphi(x)$ an eigenfunction belonging to λ_0 for this problem. The functions

$$\varphi_1(x), \varphi_2(x), \ldots, \varphi_k(x)$$

are said to be *associated with the eigenfunction* $\varphi(x)$ if all the functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_k(x)$ satisfy the boundary conditions

$$\sum_{p=0}^{\mu} \frac{1}{p!} \frac{\partial^p}{\partial \lambda^p} U_\nu(\varphi_{\mu-p}) = 0, \ \varphi_0 = \varphi, \ \mu = 0,1,\ldots,k; \ \nu = 1,2,\ldots,n, \tag{9}$$

for $\lambda = \lambda_0$, and if for $\lambda = \lambda_0$ the following relations hold:

$$\left.\begin{aligned} l(\varphi) &= 0 \\ l(\varphi_1) + \frac{1}{1!}\frac{\partial}{\partial\lambda} l(\varphi) &= 0 \\ \cdot \qquad \cdot \qquad & \quad \cdot \\ l(\varphi_k) + \frac{1}{1!}\frac{\partial}{\partial\lambda} l(\varphi_{k-1}) + \cdots + \frac{1}{k!}\frac{\partial^k}{\partial\lambda^k} l(\varphi) &= 0 \end{aligned}\right\}. \tag{10}$$

Here $\dfrac{\partial^q}{\partial\lambda^q}\, l(\psi)$ denotes the differential expression whose coefficients are the qth derivatives with respect to λ of the corresponding coefficients of the differential expression $l(\psi)$. The number k is called the *length of the system of associated functions*. An eigenfunction $\varphi(x)$ is said to have *multiplicity m* if there is a system of functions associated with $\varphi(x)$ of length $(m - 1)$ but no system of length m.

V. *If λ_0 is a ν-fold zero of the function $\Delta(\lambda)$, then the multiplicity of any eigenfunction is not greater than ν.*

Proof. We assume the contrary; there is a system of functions associated with $\varphi(x)$ of length ν:

$$\varphi_1(x),\varphi_2(x),\ldots,\varphi_\nu(x).$$

Let $y_1(x,\lambda)$ be a solution of the equation $l(y) = 0$, which, for $x = a$, satisfies the same initial conditions as the function

$$\psi(x,\lambda) = \varphi(x) + (\lambda - \lambda_0)\varphi_1(x) + (\lambda - \lambda_0)^2\varphi_2(x) + \ldots + (\lambda - \lambda_0)^\nu\varphi_\nu(x).$$

Then the first $(\nu + 1)$ terms in the expansion of the function $y_1(x,\lambda)$ in powers of $(\lambda - \lambda_0)$ agree with the corresponding addends in the representation of the function $\psi(x,\lambda)$. For, if we put $\lambda = \lambda_0$ in the relations

$$\left.\begin{aligned} &l(y_1) = 0 \\ &y_1^{(q)}(a,\lambda) = \psi^{(q)}(a,\lambda), \qquad q = 0,1,\ldots,(n-1) \end{aligned}\right\}, \tag{11}$$

it follows that the function $y_1(x,\lambda_0)$ satisfies the same equation and the same initial conditions for $x = a$ as the function $\varphi(x)$ does; hence $y_1(x,\lambda) = \varphi(x)$. If we differentiate the relations (11) with respect to λ, and then put $\lambda = \lambda_0$, we get, for $\lambda = \lambda_0$,

$$l\left(\frac{\partial y_1}{\partial\lambda}\right) + \frac{\partial}{\partial\lambda}\, l(\varphi) = 0,$$

$$\frac{\partial y_1^{(q)}(a,\lambda)}{\partial\lambda} = \varphi_1^{(q)}(a), \qquad q = 0,1,\ldots,(n-1),$$

and comparison with (10) shows that $\left[\dfrac{\partial y_1(x,\lambda)}{\partial\lambda}\right]_{\lambda=\lambda_0} = \varphi_1(x)$.

Repeating the same argument, we arrive at

$$\left[\frac{1}{q!}\frac{\partial^q y_1(x,\lambda)}{\partial\lambda^q}\right]_{\lambda=\lambda_0} = \varphi_q(x), \qquad q = 1,2,\ldots,\nu.$$

Now we construct solutions $y_2(x,\lambda),y_3(x,\lambda),\ldots,y_n(x,\lambda)$ of the equation $l(y) = 0$ such that the functions

$$y_1(x,\lambda),y_2(x,\lambda),\ldots,y_n(x,\lambda)$$

are linearly independent for $\lambda = \lambda_0$, and with the help of these functions we form the determinant $\Delta(\lambda)$.

Since, by hypothesis, the functions $\varphi,\varphi_1,\ldots,\varphi_\nu$ satisfy the boundary conditions (9), the power series for $U_\nu(y_1)$ in powers of $(\lambda - \lambda_0)$ must begin with $(\lambda - \lambda_0)^{\nu+1}$. The first column of the determinant $\Delta(\lambda)$ would therefore contain the common factor $(\lambda - \lambda_0)^{\nu+1}$, i.e. the number λ_0 would be a zero of $\Delta(\lambda)$ with at least $(\nu + 1)$-fold multiplicity, and this contradicts our hypothesis. Hence the theorem.

Now let φ_1 be an eigenfunction of maximum multiplicity, φ_2 an eigenfunction of maximum multiplicity among the eigenfunctions linearly independent of φ_1, and, in general, φ_j an eigenfunction of maximum multiplicity among the eigenfunctions linearly independent of $\varphi_1,\varphi_2,\ldots,\varphi_{j-1}$. Clearly, the total number of such eigenfunctions is equal to the multiplicity p of the eigenvalue λ_0. We denote the multiplicity of the eigenfunction φ_j by m_j, and the system of associated functions belonging to it by

$$\varphi_{j1},\varphi_{j2},\ldots,\varphi_{j,m_j-1}.$$

It follows immediately from the definition of the numbers m_j that

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_p.$$

The system of functions so obtained

$$\varphi_j,\varphi_{j1},\varphi_{j2},\ldots,\varphi_{j,m_j-1}, \qquad j = 1,2,\ldots,p,$$

is called *a canonical system of eigenfunctions and their associated functions.*

VI. *The sum of the multiplicities $m_1 + m_2 + \ldots + m_p$ is equal to the multiplicity of λ_0 as a zero of $\Delta(\lambda)$:*

i.e. $$m_1 + m_2 + \ldots + m_p = \nu.$$

Proof. Let

$$\psi_{p+1}(x),\psi_{p+2}(x),\ldots,\psi_n(x)$$

be any solutions of the equation $l(y) = 0$ for $\lambda = \lambda_0$, which together with the eigenfunctions $\varphi_1,\varphi_2,\ldots,\varphi_p$ form a fundamental system of solutions of the equation $l(y) = 0$ for $\lambda = \lambda_0$. As above, we can construct the solutions $y_1,y_2,\ldots,y_n$ of the equation $l(y) = 0$ whose expansions in powers of $(\lambda - \lambda_0)$ have the form

$$\left.\begin{aligned} y_i(x,\lambda) &= \varphi_i(x) + \varphi_{i1}(x)(\lambda - \lambda_0) + \ldots + \varphi_{i,m_i-1}(x)(\lambda - \lambda_0)^{m_i-1} \\ &\qquad + \psi_i(x)(\lambda - \lambda_0)^{m_i} + \ldots, \text{ for } i = 1,2,\ldots,p, \\ y_i(x,\lambda) &= \psi_i(x) + \ldots \qquad\qquad\qquad \text{for } i = (p + 1),\ldots,n, \end{aligned}\right\} \quad (12)$$

and these functions represent a fundamental system of solutions of the equation $l(y) = 0$ for $\lambda = \lambda_0$. We form with them the determinant $\Delta(\lambda)$. The first p columns of $\Delta(\lambda)$ contain respectively the factors $(\lambda - \lambda_0)^{m_1},(\lambda - \lambda_0)^{m_2},\ldots,$

$(\lambda - \lambda_0)^{m_p}$, and so the determinant $\Delta(\lambda)$ contains the factor $(\lambda - \lambda_0)^{m_1+m_2+\dots+m_p}$. Therefore

$$\Delta(\lambda) = (\lambda - \lambda_0)^{m_1+m_2+\dots+m_p}\Delta_1(\lambda),$$

and $\Delta_1(\lambda_0) = \det[U_{ij}]$, $i,j = 1,2,\dots,n$, with $U_{ij} = U_i(\psi_j)$ for $i = 1,\dots,n$ and $j = (p+1),(p+2),\dots,n$,

$$\text{but } U_{ij} = U_i(\psi_j) + \frac{1}{1!}\frac{\partial}{\partial\lambda}U_i(\varphi_{j,m_j-1}) + \frac{1}{2!}\frac{\partial^2}{\partial\lambda^2}U_i(\varphi_{j,m_j-2}) + \dots$$

$$+ \frac{1}{m_j!}\frac{\partial^{m_j}}{\partial\lambda^{m_j}}U_i(\varphi_j) \text{ for } i = 1,\dots,n \text{ and } j = 1,\dots,p.$$

To prove that $\nu = m_1 + m_2 + \dots + m_p$, we have only to show that $\Delta_1(\lambda_0) \neq 0$. Let us assume the contrary, that $\Delta_1(\lambda_0) = 0$. Then one column of this determinant, the column with suffix j for example, is a linear combination of the columns with suffixes greater than j; let $\alpha_{j+1},\dots,\alpha_n$ be the coefficients of this linear combination.

First, let $j > p$. Then this implies that

$$\psi = \psi_j - \alpha_{j+1}\psi_{j+1} - \dots - \alpha_n\psi_n$$

satisfies the boundary conditions $U_\nu(\psi) = 0$, $\nu = 1,\dots,n$, for $\lambda = \lambda_0$. Since in this case all the $\psi_j,\psi_{j+1},\dots,\psi_n$ are solutions of $l(y) = 0$, ψ would be an eigenfunction and would have to be representable as a linear combination of the functions $\varphi_1,\varphi_2,\dots,\varphi_p$. But this contradicts the linear independence of the system $\varphi_1,\varphi_2,\dots,\varphi_p,\psi_{p+1},\dots,\psi_n$.

If, on the other hand, $j < p$, then we put

$$\tilde{y}_i(x,\lambda) = (\lambda - \lambda_0)^{m_j - m_i}y_i(x,\lambda) \quad \text{for } i \leqslant p,$$

$$\tilde{y}_i(x,\lambda) = (\lambda - \lambda_0)^{m_j}y_i(x,\lambda) \quad \text{for } i > p,$$

and

$$y(x,\lambda) = \tilde{y}_j(x,\lambda) - \alpha_{j+1}\tilde{y}_{j+1}(x,\lambda) - \dots - \alpha_n\tilde{y}_n(x,\lambda).$$

Then

$$l(y) = 0.$$

By differentiating this equation m_j times with respect to λ and then putting $\lambda = \lambda_0$. it follows that the functions

$$\frac{1}{1!}\frac{\partial y(x,\lambda)}{\partial\lambda}, \frac{1}{2!}\frac{\partial^2 y(x,\lambda)}{\partial\lambda^2}, \dots, \frac{1}{m!}\frac{\partial^{m_j} y(x,\lambda)}{\partial\lambda^{m_j}} = \psi(x)$$

satisfy the equations (10) with $k = m_j$. On the other hand, it also follows that the boundary conditions (9) are satisfied by the same functions, again with $k = m_j$, and in particular the last equation

$$\sum_{p=0}^{m_j}\frac{1}{p!}\frac{\partial^p U_\nu}{\partial\lambda^p}\left(\frac{1}{(m_j - p)!}\frac{\partial^{m_j - p}y}{\partial\lambda^{m_j - p}}\right) = 0$$

holds, again because of the vanishing of $\Delta_1(\lambda_0)$ and the special construction of

$y(x,\lambda)$. These functions would therefore form a system of functions (of length m_j) associated with the eigenfunction $y(x,\lambda_0)$. But this is impossible, because $y(x,\lambda_0)$ is a linear combination of the eigenfunctions $\varphi_j,\varphi_{j+1},\ldots,\varphi_p$, and therefore, by the definition of these functions, it has a multiplicity of at most m_j.

Theorem VI means that a canonical system of eigenfunctions and their conjugate functions generate a subspace, the dimension of which is equal to the multiplicity of the zero of $\Delta(\lambda)$ (for it can readily be seen that the eigenfunctions and associated functions of the canonical system are linearly independent), so that the associated functions complete the set of eigenfunctions to form a space of the necessary dimension.

The theory of associated functions is analogous to the theory of elementary divisors of linear operators in a finite-dimensional space (cf., e.g. [28a]). The presentation of the theory of associated functions given here was originated by Keldysh [47].

4. Relation between the Eigenvalues and the Eigenfunctions of Adjoint Operators[1]

THEOREM 1

If λ is an eigenvalue of multiplicity p of the operator L, then $\bar{\lambda}$ is an eigenvalue of the adjoint operator L^* and has the same multiplicity p.

Proof. The operator L is generated by the differential expression $l(y)$ and the boundary conditions

$$U_\nu(y) = 0,\ \nu = 1,2,\ldots,n,$$

while the operator L^* is generated by the differential expression $l^*(y)$ and the conditions

$$V_\nu(y) = 0,\ \nu = 1,2,\ldots,n.$$

We put

$$l_1(y) = l(y) - \lambda y;$$

then

$$l_1^*(y) = l^*(y) - \bar{\lambda} y.$$

If λ is an eigenvalue of multiplicity p of the operator L, then the boundary-value problem

$$l_1(y) = 0, \qquad U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n$$

has p linearly independent solutions.

By the results of §1.7 (Theorem VII) it follows that the adjoint boundary-value problem

$$l_1^*(y) = 0, \qquad V_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n,$$

also has p linearly independent solutions. This means that $\bar{\lambda}$ is an eigenvalue of the operator L^* and has multiplicity p.

Two functions $y(x)$ and $z(x)$ are said to be *orthogonal* if $(y,z) = 0$.[2]

---

[1] We remind the reader that only the case $m = n$ is now being treated.

[2] We remind the reader of the notation introduced in §1: $(y,z) = \int_a^b y(x).\overline{z(x)}dx$.

THEOREM 2

Eigenfunctions of the operators L and L^* corresponding to the eigenvalues λ and μ respectively are orthogonal if $\lambda \neq \bar{\mu}$.

Proof. Let y be an eigenfunction of the operator L for the eigenvalue λ, and z an eigenfunction of the operator L^* for the eigenvalue μ, so that

$$Ly = \lambda y$$

and

$$L^*z = \mu z.$$

Hence

$$(Ly,z) = (\lambda y,z) = \lambda(y,z),$$

and

$$(y,L^*z) = (y,\mu z) = \bar{\mu}(y,z).$$

But from §1.6 (equation (20)), $(Ly,z) = (y,L^*z)$; so on subtracting the last two equations we obtain

$$(\lambda - \bar{\mu})(y,z) = 0.$$

If $\lambda \neq \bar{\mu}$ this implies $(y,z) = 0$, i.e. the functions y and z are orthogonal.

5. Eigenvalues and Eigenfunctions of a Self-adjoint Operator

THEOREM 3

All eigenvalues of a self-adjoint operator are real.

Proof. For a self-adjoint operator L we have

$$(Ly,z) = (y,Lz);$$

in particular,

$$(Ly,y) = (y,Ly).$$

But $(y,Ly) = \overline{(Ly,y)}$; hence (Ly,y) is real.

Now let λ be an eigenvalue of the operator L, and y an eigenfunction belonging to λ. The equation $Ly = \lambda y$ implies that $(Ly,y) = \lambda(y,y)$.

Since $(y,y) > 0$ and (Ly,y) is real, $\lambda = \dfrac{(Ly,y)}{(y,y)}$ must also be real.

Theorems 2 and 3 yield the

COROLLARY

Eigenfunctions of a self-adjoint operator which belong to different eigenvalues are orthogonal.

Eigenfunctions which belong to one and the same eigenvalue can also be chosen so as to be mutually orthogonal. To do this, we simply choose an orthogonal basis in the subspace of eigenfunctions which belong to one and the same eigenvalue.

Altogether, we obtain in this way a system of pairwise orthogonal eigenfunctions of the self-adjoint operator L.

6. Examples of Eigenvalue Problems

1. $$-y'' = \lambda y; \qquad y(0) = y(1); \qquad y'(0) = y'(1). \tag{13}$$

The general solution of the differential equation is

$$y = A \cos x\sqrt{\lambda} + B \sin x\sqrt{\lambda}.$$

By substitution into the boundary conditions, we see at once that the eigenvalues are given by

$$\lambda_n = (2n\pi)^2, \qquad n = 0,1,2,3,\ldots$$

For $n \neq 0$, there are, corresponding to the eigenvalue λ_0, two linearly independent eigenfunctions

$$\cos(2n\pi x) \text{ and } \sin(2n\pi x);$$

λ_n is therefore a double eigenvalue for $n \neq 0$.

For $n = 0$, there is, corresponding to the eigenvalue λ_0, only one eigenfunction $y = 1$ (up to a constant factor); λ_0 is therefore a simple eigenvalue.

2. *Flexure of a Rod with one end fixed and one end free.*

We consider a rod with one end, $x = l$, firmly fixed (Fig. 1) and its other end free, $x = 0$. At the free end a compressive force P acts along the rod.

It is a familiar fact that for sufficiently small values of P the straight line shape of the rod is stable, but there is a critical value P_0 of the force P such that, for $P > P_0$, the rod loses its straightness and bends.

We shall investigate this bending by considering an equilibrium position differing but little from the straight-line position (Fig. 2).

The equation for the bent axis of the rod $y = y(x)$ is then such that

$$Py = -EIy'', \tag{14}$$

where I is the moment of inertia of the cross-section of the rod and E is the modulus of elasticity. The two sides of this equation represent the bending moment due to the applied force and the bending moment in the rod at distance x from the origin (see [111], for instance).

We restrict ourselves to the simple case of a homogeneous rod of constant cross-section; then the stiffness EI is constant. Using the substitution $\lambda = \dfrac{P}{EI}$, we obtain the equation

$$-y'' = \lambda y. \tag{15}$$

From Fig. 2 it is clear that the function $y(x)$ satisfies the conditions

$$y(0) = 0, \qquad y'(l) = 0. \tag{16}$$

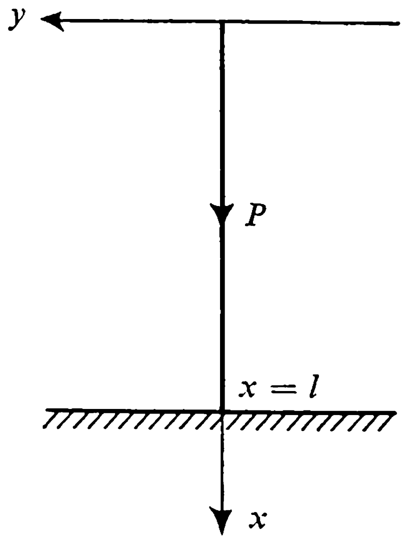

Fig. 1.

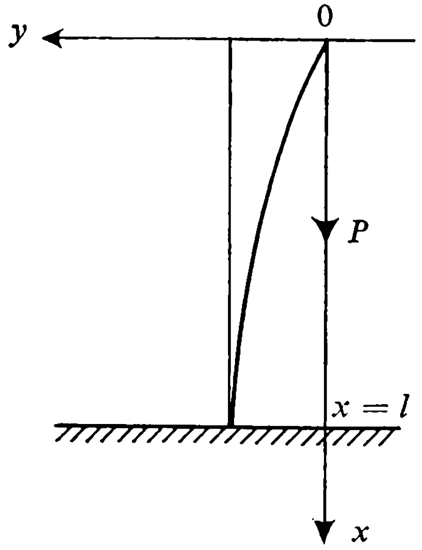

Fig. 2.

So, by (15) and (16) we have formulated the eigenvalue problem. By substituting the general solution

$$y = A\cos x\sqrt{\lambda} + B\sin x\sqrt{\lambda}$$

for equation (15) into the boundary conditions (16) we find $A = 0$, $\cos\sqrt{\lambda}l = 0$, and so

$$\lambda = \left(\frac{2n-1}{2l}\pi\right)^2, \qquad n = 1,2,3,\ldots$$

These eigenvalues are simple, and to them correspond the eigenfunctions

$$y(x) = B\sin\frac{2n-1}{2l}\pi x.$$

The critical loads corresponding to the eigenvalues are

$$P_n = EI\left(\frac{2n-1}{2l}\right)^2.$$

Apart from a numerical factor, the eigenfunctions describe the corresponding equilibrium forms of the rod (Fig. 3).

For a rod with varying cross-section or for a non-homogeneous rod, EI is a function of x. Writing $\frac{1}{EI} = \rho(x)$, we obtain the following eigenvalue problem:

$$-y'' = P\rho(x)y, \qquad y(0) = 0, \qquad y'(l) = 0.$$

In the general case for an arbitrary function $\rho(x)$, only approximate methods are possible for calculating the eigenvalues and eigenfunctions.

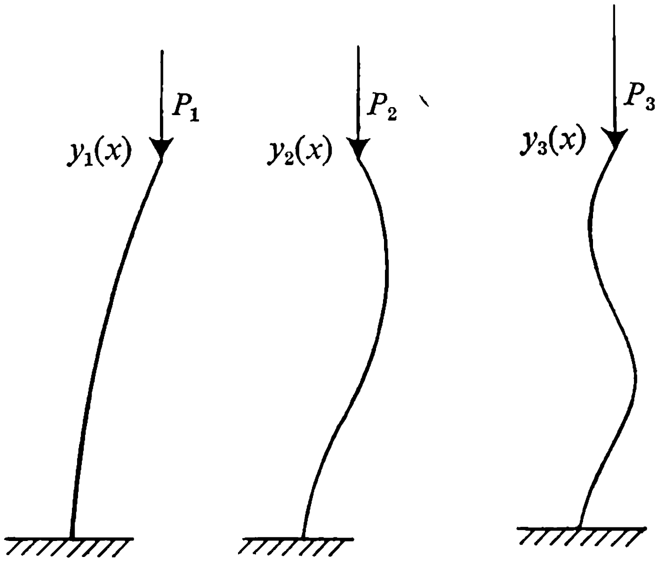

Fig. 3.

3. *Flexure of a Rod with one end fixed and one end supported.*

In this case there is a horizontal reaction H at the end $x = 0$ (Fig. 4). So the bending moment is $M = Py - Hx = -EIy''$.

By differentiating this equation twice, we get

$$(EIy'')'' = -Py''; \tag{17}$$

and the function $y(x)$ must satisfy the boundary conditions

$$y(0) = 0, \quad y''(0) = 0, \quad y(l) = 0, \quad y'(l) = 0. \tag{18}$$

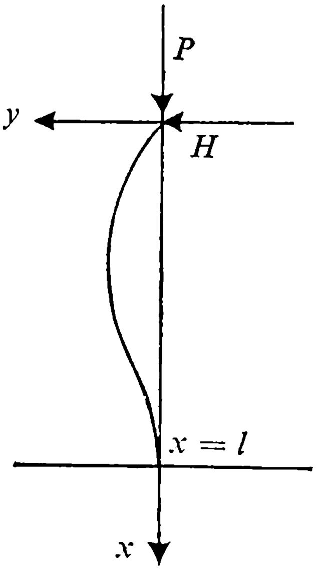

Fig. 4.

The first and third of these boundary conditions are obvious; the second means that the bending moment vanishes at the point $x = 0$ (since the rod is supported there); the fourth holds because the end $x = l$ is fixed and therefore the tangent there is parallel to the x-axis.

So we have the eigenvalue problem (17), (18). Suppose now that EI is constant; we write $P/EI = k^2$, and also, for simplicity, take $l = 1$. Then equation (17) can be written in the form

$$y'''' = -k^2 y''. \tag{19}$$

Its general integral, for $k \neq 0$, is

$$y = A + Bx + C\cos kx + D\sin kx. \tag{20}$$

The boundary conditions (18) give directly that $A = C = 0$ and that the eigenvalues are the roots of the equation

$$\tan k = k.$$

These are the abscissae of the points of intersection of the graphs (Fig. 5)

$$y = \tan x \text{ and } y = x.$$

The eigenvalues so obtained are simple, and the corresponding eigenfunctions are

$$y = \sin kx - x\sin k.$$

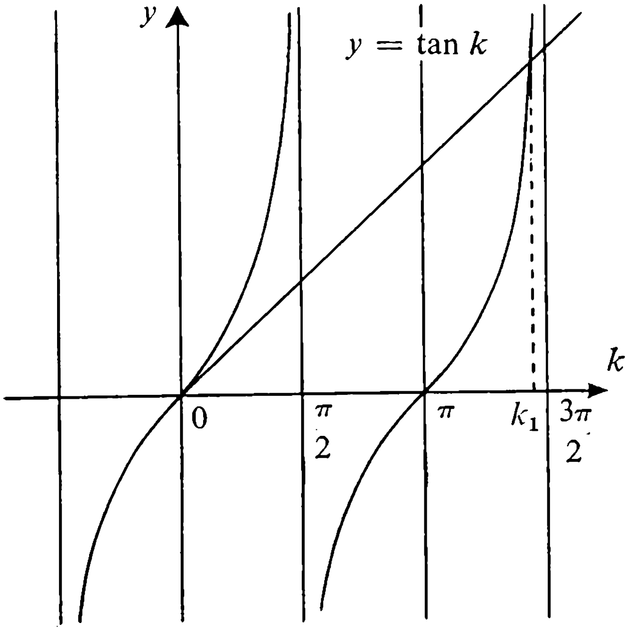

Fig. 5.

For $k = 0$, the general integral of the differential equation is

$$y = A + Bx + Cx^2 + Dx^3; \tag{21}$$

but the boundary conditions (18) can be satisfied only if $A = B = C = D = 0$. Consequently $k = 0$ is not an eigenvalue of the problem.

4. $y'''' = -k^2y''; \qquad y''(0) = y'''(0) = y(1) = y'(1) = 0.$

For $k = 0$, the general integral (21) of the differential equation satisfies the given boundary conditions only if $A = B = C = D = 0$. If $k \neq 0$, the general integral (20) of the differential equations satisfies these boundary conditions again only if $A = B = C = D = 0$. Consequently *this problem has no eigenvalues.*

5. $y'''' = -k^2y''; \qquad y(0) = y'(0) = y''(0) = y(1) = 0.$

As above, we find that $k = 0$ is not an eigenvalue. If, for $k \neq 0$, we substitute the general solution (20) of the differential equation into the boundary conditions, we obtain

$$\sin k = k.$$

This equation has no real roots other than zero. So *the problem considered has only non-real eigenvalues.*

6. *Natural Vibrations of a Stretched String.*

The equation of free, transverse vibrations of a string has the form

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{22}$$

where $u = u(x,t)$ is the displacement of the point of the string with abscissa x at time t, and a is a constant.

Solutions of equation (22) which are of the form

$$y = y(x) \sin (a\omega t + \varphi) \tag{23}$$

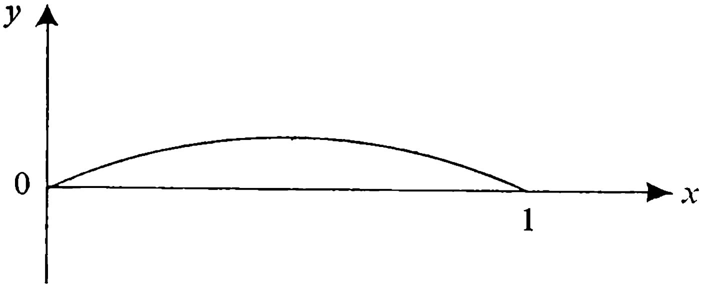

Fig. 6.

represent the natural vibrations of the string. Substitution in (22) shows that the function $y(x)$ has to satisfy the differential equation

$$y'' + \omega^2 y = 0. \tag{24}$$

If we take the length of the string as unity, then the condition that the string is fixed at its ends (Fig. 6) can be written in the form

$$y(0) = y(1) = 0. \tag{25}$$

So we have an eigenvalue problem (24), (25).

We substitute the general solution

$$y = A \cos \omega x + B \sin \omega x$$

of equation (24) into the boundary conditions (25) and obtain $A = 0$, $\sin \omega = 0$.

Hence

$$\omega = n\pi, \qquad n = 1,2,3,\dots .$$

So

$$\omega_n^2 = (n\pi)^2, \qquad n = 1,2,3,\dots$$

are the eigenvalues. They are simple, and to them belong the eigenfunctions

$$y_n(x) = B_n \sin n\pi x.$$

The eigenvalues and eigenfunctions have a simple interpretation in mechanics. Up to a constant factor, they are the squares of the natural frequencies of the string and the eigenfunctions yield the modal shapes for the natural vibrations.

§3. *Green's Function for a Linear Differential Operator*

1. General Definition of the Inverse Operator

An operator B is said to be *inverse to an operator* A if the domain of definition $\mathscr{D}_B$ of the operator B coincides with the range of values $\mathscr{R}_A$ of the operator A and if, for all $x \in \mathscr{D}_A$, the relation

$$B(Ax) = x$$

holds. The inverse operator to A will be denoted by A^{-1}. Hence, for all $x \in \mathscr{D}_A$,

$$A^{-1}(Ax) = x \tag{1}$$

I. *If the operator A has an inverse operator A^{-1}, then the operator A is inverse to the operator A^{-1}*: i.e.

$$(A^{-1})^{-1} = A. \tag{2}$$

For equation (1) shows that the range of values $\mathscr{R}_{A^{-1}}$ of the operator A^{-1}

coincides with the domain of definition $\mathscr{D}_A$ of the operator A. Further, if we apply the operator A to both sides of (1) and put $y = Ax$, we get

$$AA^{-1}y = y$$

for all $y \in \mathscr{R}_A = \mathscr{D}_{A^{-1}}$. And this implies precisely that A is inverse to A^{-1}.

II. *The operator A has an inverse if and only if the equation*

$$Ax = 0 \tag{3}$$

has only the "trivial" solution $x = 0$.

For, suppose that the operator A has an inverse operator A^{-1}. If we apply A^{-1} to both sides of (3), we obtain $x = 0$. Hence the condition (3) is necessary.

If, on the other hand, we assume that the condition (3) holds, then the equation

$$Ax = y, \tag{4}$$

for all $y \in \mathscr{R}_A$, has a unique solution $x \in \mathscr{D}_A$; because, for each such y, at least one solution exists by virtue of the definition of the range $\mathscr{R}_A$. And the solution is unique; for if $Ax' = y$ were also true, then we should have $A(x - x') = 0$. By hypothesis, this is possible only if $x - x' = 0$, i.e. if $x = x'$. To each element $y \in \mathscr{R}_A$ we assign the solution x of equation (4). But we see immediately that this correspondence is a linear operator, and moreover it is the inverse to A.

2. The Problem of Inverting a Differential Operator

Let L be a differential operator, generated from the differential expression

$$l(y) = p_0(x)\frac{d^ny}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + p_n(x)y$$

and the boundary conditions

$$U_\nu(y) = 0, \qquad \nu = 1,2,\dots,n.$$

We assume that the homogeneous boundary-value problem $Ly = 0$ has only the trivial solution $y = 0$; i.e. for an arbitrary fundamental system of solutions $y_1,y_2,\dots,y_n$ of the equation $l(y) = 0$, the rank of the matrix

$$\begin{bmatrix} U_1(y_1) & U_1(y_2) & \dots & U_1(y_n) \\ U_2(y_1) & U_2(y_2) & \dots & U_2(y_n) \\ \cdot & \cdot & & \cdot \\ U_n(y_1) & U_n(y_2) & \dots & U_n(y_n) \end{bmatrix}$$

is equal to n, and so the determinant

$$\det |U_i(y_j)| \neq 0, \qquad i,j = 1,2,\dots,n. \tag{5}$$

By II, L then has an inverse L^{-1}, whose domain of definition coincides with the range of values of the operator L. We set ourselves the task of finding an explicit expression for L^{-1}. As we shall see, L^{-1} turns out to be an integral operator with a continuous kernel; the kernel is called *Green's function* for the operator L. We shall give below an explicit expression for the Green's function.

3. Construction of Green's Function

Green's function for an operator L is to be understood to be a function $G(x,\xi)$ satisfying the following conditions:

1°. *$G(x,\xi)$ is continuous and has continuous derivatives with respect to x up to order $(n-2)$ inclusive for all values of x and ξ in the interval $[a,b]$.*

2°. *For any fixed value of ξ in the interval (a,b) the function $G(x,\xi)$ has continuous derivatives of orders $(n-1)$ and n with respect to x in each of the intervals $[a,\xi)$ and $(\xi,b]$; the $(n-1)$th derivative is discontinuous at $x=\xi$ with a jump of $\frac{1}{p_0(\xi)}$:*

$$\frac{\partial^{n-1}}{\partial x^{n-1}} G(\xi+0,\xi) - \frac{\partial^{n-1}}{\partial x^{n-1}} G(\xi-0,\xi) = \frac{1}{p_0(\xi)}.$$

3°. *In each of the intervals $[a,\xi)$ and $(\xi,b]$, $G(x,\xi)$, considered as a function of x, satisfies the equation $l(G)=0$ and the boundary conditions $U_\nu(G)=0$, $\nu=1,2,\ldots,n$.*

THEOREM 1

If the boundary-value problem $Ly=0$ has only the trivial solution, then the operator L has one and only one Green's function.

Proof. Let $y_1,y_2,\ldots,y_n$ be linearly independent solutions of the equation $l(y)=0$. Since the function G in the interval $[a,\xi)$ also satisfies this equation, we have

$$G(x,\xi) = a_1y_1(x) + \ldots + a_ny_n(x) \qquad \text{for } a \leqslant x < \xi,$$

where $a_1,a_2,\ldots,a_n$ are certain functions of ξ. Similarly,

$$G(x,\xi) = b_1y_1(\mathrm{x}) + \ldots + b_ny_n(\mathrm{x}) \qquad \text{for } \xi < x \leqslant b.$$

The requirement for continuity of the function $G(x,\xi)$ and its first $(n-2)$ derivatives at $x=\xi$ gives the equations

$$\begin{aligned}
&[a_1y_1(\xi) + \ldots + a_ny_n(\xi)] - [b_1y_1(\xi) + \ldots + b_ny_n(\xi)] = 0,\\
&[a_1y_1'(\xi) + \ldots + a_ny_n'(\xi)] - (b_1y_1'(\xi) + \ldots + b_ny_n'(\xi)] = 0,\\
&\quad . \quad \ldots \quad . \qquad\qquad . \quad \ldots \quad .\\
&[a_1y_1^{(n-2)}(\xi) + \ldots + a_ny_n^{(n-2)}(\xi)] - [b_1y_1^{(n-1)}(\xi) + \ldots + b_ny_n^{(n-2)}(\xi)] = 0.
\end{aligned}$$

Moreover, the condition

$$\frac{\partial^{n-1}}{\partial x^{n-1}} G(\xi + 0,\xi) - \frac{\partial^{n-1}}{\partial x^{n-1}} G(\xi - 0,\xi) = \frac{1}{p_0(\xi)}$$

is equivalent to

$$[a_1 y_1^{(n-1)}(\xi) + \dots + a_n y_n^{(n-1)}(\xi)] - [b_1 y_1^{(n-1)}(\xi) + \dots + b_n y_n^{(n-1)}(\xi)] = -\frac{1}{p_0(\xi)}.$$

Putting

$$c_\nu = b_\nu - a_\nu, \qquad \nu = 1,2,\dots,n,$$

we obtain for the c_ν the system of equations

$$\left.\begin{aligned} c_1 y_1(\xi) + \dots + c_n y_n(\xi) \quad &= 0, \\ c_1 y_1'(\xi) + \dots + c_n y_n'(\xi) \quad &= 0, \\ \cdot \quad \cdots \quad \cdot \quad &\quad \cdot \\ c_1 y_1^{(n-2)}(\xi) + \dots + c_n y_n^{(n-2)}(\xi) &= 0, \\ c_1 y_1^{(n-1)}(\xi) + \dots + c_n y_n^{(n-1)}(\xi) &= +\frac{1}{p_0(\xi)}, \end{aligned}\right\} \tag{6}$$

the determinant of which is the Wronskian determinant for the fundamental system $y_1,\dots,y_n$ for $x = \xi$, and therefore does not vanish.

Hence the system always has a solution and the functions c_ν are uniquely determined by the system (6). To determine the functions a_ν and b_ν we bring in the boundary conditions; we write the form $U_\nu(y)$ as

$$U_\nu(y) = U_{\nu a}(y) + U_{\nu b}(y), \tag{7}$$

where $U_{\nu a}(y)$ is the sum of all the addends containing $y_a, y_a', \dots, y_a^{(n-1)}$, and $U_{\nu b}(y)$ is the sum of all the addends containing $y_b, y_b', \dots, y_b^{(n-1)}$.

Hence

$$U_\nu(G) = a_1 U_{\nu a}(y_1) + \dots + a_n U_{\nu a}(y_n) + b_1 U_{\nu b}(y_1) + \dots + b_n U_{\nu b}(y_n) = 0.$$

Replacing the coefficients a_k by their expressions as $a_k = b_k - c_k$, we get

$$b_1 U_{\nu b}(y_1) + \dots + b_n U_{\nu b}(y_n) + (b_1 - c_1) U_{\nu a}(y_1) + \dots + (b_n - c_n) U_{\nu a}(y_n) = 0.$$

Hence, by (7),

$$b_1 U_\nu(y_1) + \dots + b_n U_\nu(y_n) = c_1 U_{\nu a}(y_1) + \dots + c_n U_{\nu a}(y_n). \tag{8}$$

Putting $\nu = 1,2,\dots,n$, we obtain from (8) a system of equations in the unknowns $b_1, b_2, \dots, b_n$, for which, by (5), the determinant does not vanish. Consequently the system of equations has a unique solution $b_1, b_2, \dots, b_n$. But then the functions a_ν are also determined uniquely from $a_\nu = b_\nu - c_\nu$. So we have proved the existence and the uniqueness of the Green's function.

4. Inversion of a Differential Operator by means of the Green's Function

Suppose that the equation $Ly = 0$ has only the trivial solution $y = 0$, so that the inverse L^{-1} (see subsection 1) and its Green's function (defined as in subsection 3) exist. If $L^{-1}f = y$, then

$$Ly = f, \tag{9}$$

i.e. y is a solution of the equation

$$l(y) = f, \tag{10}$$

and also satisfies the boundary conditions

$$U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n. \tag{11}$$

We shall show that this solution exists for any function $f(x)$ which is continuous in the interval $[a,b]$, and that it can be determined by means of the Green's function. More precisely, the following theorem is valid.

THEOREM 2

If the equation $Ly = 0$ *has only the trivial solution, then, for any function* $f(x)$ *which is continuous in the interval* $[a,b]$, *there exists a solution of the equation* $Ly = f$; *this solution is expressed by the formula*

$$y(x) = \int_a^b G(x,\xi)f(\xi)d\xi, \tag{12}$$

where $G(x,\xi)$ *denotes the Green's function for the operator* L.

Proof. Let $y(x)$ be the function defined by formula (12). We show that $y(x)$ satisfies the conditions (10) and (11), so that it is a solution of the equation $Ly = f$.

The function $G(x,\xi)$ has continuous derivatives up to the $(n - 2)$-th order inclusive; so we may differentiate with respect to x under the integral sign in (12) $(n - 2)$ times. Hence

$$y^{(\nu)}(x) = \int_a^b \frac{\partial^\nu G(x,\xi)}{\partial x^\nu} f(\xi)d\xi, \qquad \nu = 1,2,\ldots,(n-2). \tag{13}$$

Hence the function $y(x)$ and its derivatives $y^{(\nu)}(x)$ up to the $(n - 2)$-th order inclusive are continuous in the interval $[a,b]$.

The function $\frac{\partial^{n-1}G}{\partial x^{n-1}}$, on the other hand, has a discontinuity at $x = \xi$. So in calculating $y^{(n-1)}$ and $y^{(n)}$ we may not differentiate any more under the integral sign without preliminary manipulation. For this reason we write formula (13) for $\nu = n - 2$ in the following form:

$$y^{(n-2)}(x) = \int_a^x \frac{\partial^{n-2}G(x,\xi)}{\partial x^{n-2}} f(\xi)d\xi + \int_x^b \frac{\partial^{n-2}G(x,\xi)}{\partial x^{n-2}} f(\xi)d\xi.$$

In each of the intervals (a,x) and (x,b) the integrand and its derivative with respect to x are continuous; we therefore differentiate with respect to x under the integral sign and with respect to the upper (or lower) limit x, and obtain:

$$y^{(n-1)}(x) = \int_a^x \frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} f(\xi)d\xi + \left[\frac{\partial^{n-2}G(x,\xi)}{\partial x^{n-2}}\right]_{\xi=x-0} f(x)$$
$$+ \int_x^b \frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} f(\xi)d\xi - \left[\frac{\partial^{n-2}G(x,\xi)}{\partial x^{n-2}}\right]_{\xi=x+0} f(x) \quad (14)$$

Since $\frac{\partial^{n-2}G}{\partial x^{n-2}}$ is continuous at $x = \xi$, the two integrated terms cancel out, and there remains

$$y^{(n-1)}(x) = \int_a^x \frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} f(\xi)d\xi + \int_x^b \frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} f(\xi)d\xi, \quad (15)$$

i.e.
$$y^{(n-1)}(x) = \int_a^b \frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} f(\xi)d\xi. \quad (16)$$

By differentiating the formula (15) again we find, as above:

$$y^{(n)}(x) = \int_a^x \frac{\partial^{n}G(x,\xi)}{\partial x^{n}} f(\xi)d\xi + \left[\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}}\right]_{\xi=x-0} f(x)$$
$$+ \int_x^b \frac{\partial^{n}G(x,\xi)}{\partial x^{n}} f(\xi)d\xi - \left[\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}}\right]_{\xi=x+0} f(x). \quad (17)$$

From the condition 2° in the definition of Green's function,

$$\left[\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}}\right]_{\xi=x-0} - \left[\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}}\right]_{\xi=x+0} = \frac{1}{p_0(x)}.$$

Hence (17) can be written in the form

$$y^{(n)}(x) = \int_a^b \frac{\partial^{n}G(x,\xi)}{\partial x^{n}} f(\xi)d\xi + \frac{1}{p_0(x)} f(x). \quad (18)$$

The forms $U_\nu(y)$ contain merely the function values of $y(x)$ and its derivatives up to the order $(n - 1)$ inclusive at the points $x = a$ and $x = b$; hence, by (12), (13), and (16),

$$U_\nu(y) = \int_a^b U_\nu(G)f(\xi)d\xi = 0,$$

since, by definition of Green's function, $U_\nu(G) = 0$. So we have now shown that the function $y(x)$ satisfies the boundary conditions (11).

We shall now show that it also satisfies the equation $l(y) = f$.

By substituting the expressions (12), (13), (16) and (18) for the function $f(x)$ and its derivatives into $l(y)$, we find:

$$l(y) = \int_a^b l(G)f(\xi)d\xi + f(x).$$

But the integral in the last formula vanishes, because, by hypothesis, the function $G(x,\xi)$, regarded as a function of x, satisfies the equation $l(y) = 0$ in each of the intervals $[a,\xi)$ and $(\xi,b]$. This completes the proof of Theorem 2.

An operator A defined for all continuous functions $f(x)$ by the equation

$$Af(x) = \int_a^b k(x,\xi)f(\xi)d\xi$$

is called an *integral operator with the kernel* $k(x,\xi)$.

Theorem 2 means that *the operator* L^{-1} *is an integral operator with the kernel* $G(x,\xi)$.

5. Green's Function for the Adjoint Operator

If L has an inverse, then, by virtue of Theorem VIII, §1.7, the adjoint operator L^* also has an inverse. Consequently L^* likewise has a Green's function, which we shall denote by $H(x,\xi)$.

Let us find the relation between the Green's functions $G(x,\xi)$ and $H(x,\xi)$ of L and L^*.

Put $L^{-1}f = \varphi$, and $L^{*-1}g = \psi$, then $f = L\varphi$, $g = L^*\psi$, and the equation $(\varphi,L^*\psi) = (L\varphi,\psi)$ [see (20), §1.6] can be written in the form

$$(L^{-1}f,g) = (f,L^{*-1}g).$$

Hence

$$\int_a^b \int_a^b G(x,\xi)f(\xi)\overline{g(x)}dxd\xi = \int_a^b \int_a^b f(\xi)\overline{H(\xi,x)g(x)}dxd\xi$$

for any continuous functions $f(x)$ and $g(x)$. This gives us:

$$G(x,\xi) = \overline{H(\xi,x)}. \tag{19}$$

Kernels G and H which are related to one another by the formula (19) are said to be *adjoint* to one another.

So we have:

III. *The Green's functions of two mutually adjoint differential operators are mutually adjoint kernels.*

In particular, if the operator L is self-adjoint, so that $L^* = L$, then $H(x,\xi) = G(x,\xi)$. The relation (19) then gives the condition

$$G(x,\xi) = \overline{G(\xi,x)}. \tag{20}$$

A kernel which satisfies condition (20) is called *Hermitian*. Hence:

IV. *The Green's function of a self-adjoint differential operator is a Hermitian kernel.*

6. Boundary-value Problems involving a Parameter and their Reduction to an Integral Equation

In many questions we have to investigate boundary-value problems of the form

$$l(y) = \lambda y + f(x), \tag{21}$$

$$U_\nu(y) = 0, \qquad \nu = 1,2,\dots,n, \tag{22}$$

where λ is a certain parameter, and $f(x)$ denotes a prescribed, continuous function of x. Using the operator L generated by the differential expression $l(y)$ and the boundary conditions (22), we can write this boundary-value problem in the form

$$Ly = \lambda y + f. \tag{23}$$

For the special case $f = 0$, this equation takes the form

$$Ly = \lambda y, \tag{24}$$

and is therefore equivalent to the homogeneous boundary-value problem

$$l(y) = \lambda y; \qquad U_\nu(y) = 0, \qquad \nu = 1,2,\dots,n. \tag{25}$$

The values of λ for which equation (24) has a non-trivial solution are obviously the *eigenvalues* of the operator L (or of the boundary-value problem (25)), and each non-trivial solution itself is a related *eigenfunction.*

Suppose that L has an inverse L^{-1}; let $G(x,\xi)$ be the associated Green's function. By applying the operator L^{-1} to both sides of equation (23), we obtain

$$y = \lambda L^{-1}y + g \tag{26}$$

with $g = L^{-1}f$. Equation (26) implies

$$y(x) = \lambda \int_a^b G(x,\xi)y(\xi)d\xi + g(x). \tag{27}$$

Hence:

V. *If the differential expression $l(y)$, subject to the boundary conditions $U_\nu(y) = 0, \nu = 1,2,\dots,n$, has the Green's function $G(x,\xi)$, then the boundary-value problem*

$$l(y) = \lambda y + f(x),\ U_\nu(y) = 0$$

is equivalent to the integral equation

$$y(x) = \lambda \int_a^b G(x,\xi)y(\xi)d\xi + g(x),$$

where

$$g(x) = \int_a^b G(x,\xi)f(\xi)d\xi.$$

In particular, the homogeneous boundary-value problem

$$l(y) = \lambda y, \qquad U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n$$

is equivalent to the homogeneous integral equation

$$y(x) = \lambda \int_a^b G(x,\xi)y(\xi)d\xi. \tag{28}$$

Since $G(x,\xi)$ is a continuous kernel, the Fredholm theory (see, for example, [86a]) can be applied to the related integral equation. Accordingly, the homogeneous integral equation (28) has only denumerably many eigenvalues $\lambda_1,\lambda_2,\lambda_3,\ldots$, which have no finite point of accumulation. For all values of λ different from the eigenvalues, the inhomogeneous equation (27) with an arbitrary continuous function $g(x)$ on the right-hand side has a solution expressed by the formula

$$y(x) = \lambda \int_a^b \Gamma(x,\xi,\lambda)g(\xi)d\xi, \tag{29}$$

where $\Gamma(x,\xi,\lambda)$ denotes the *resolvent* of the kernel $G(x,\xi)$ and is a meromorphic function of λ, for arbitrary fixed values of x and ξ in the interval $[a,b]$, which can have poles only at the eigenvalues of the homogeneous integral equation. From the integral equations we go over to the corresponding boundary-value problems to obtain the following results:

VI. *If the differential equation $l(y) = 0$, under the boundary conditions $U_\nu(y) = 0$, $\nu = 1,2,\ldots,n$, has only the trivial solution, then:*

(a) *the homogeneous boundary-value problem*

$$l(y) = \lambda y\colon\ U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n,$$

has at most denumerably many eigenvalues $\lambda_1,\lambda_2,\lambda_3,\ldots$ with no finite point of accumulation.

(b) *the inhomogeneous boundary-value problem*

$$l(y) = \lambda y + f\colon\ U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n,$$

has a solution for all values of λ different from the eigenvalues for the corresponding homogeneous problem and for any continuous function $f(x)$.

We had already obtained proposition VIa otherwise, without using integral equations, in §2.1; and we could have obtained VIb by considerations similar to those there used.

7. Green's Function for the Operator $L - \lambda 1$

As before, let L be an operator generated by the expression $l(y)$ and the conditions $U_\nu(y) = 0$, $\nu = 1,2,\ldots,n$. We want to find an expression for the Green's function of the operator $L - \lambda 1$; in other words, we want to form the inverse of the operator $L - \lambda 1$.

We denote by $y_\nu = y_\nu(x,\lambda)$, $\nu = 1,2,\ldots,n$, a system of solutions of the equation $l(y) = \lambda y$ which satisfy the initial conditions

$$y_j^{(\nu-1)}(a,\lambda) = \begin{cases} 0 \text{ for } j \neq \nu, \\ 1 \text{ for } j = \nu. \end{cases}$$

We apply the method of variation of the parameters to the equation

$$l(y) - \lambda y = f,$$

and obtain

$$y(x) = \sum_{\nu=1}^{n} C'_\nu y_\nu(x) + \int_a^x f(\xi)\left(\sum_{\nu=1}^{n} \frac{W_\nu(\xi)}{W(\xi)}\right) d\xi \tag{30 a}$$

and also

$$y(x) = \sum_{\nu=1}^{n} C''_\nu y_\nu(x) - \int_x^b f(\xi)\left(\sum_{\nu=1}^{n} \frac{W_\nu(\xi)}{W(\xi)}\right) d\xi. \tag{30 b}$$

Here $C'_1,\ldots,C'_n,C''_1,\ldots,C''_n$ are constants and W is the Wronskian determinant of the functions $y_1,y_2,\ldots,y_n$:

$$W = \begin{vmatrix} y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ \cdot & \cdot & & \cdot \\ y_1 & y_2 & \cdots & y_n \end{vmatrix},$$

and $W_1, W_2,\ldots,W_n$ are the cofactors of the elements in the first row of W. Adding (30 a) and (30 b), and dividing by 2, we get

$$y(x) = \sum_{\nu=1}^{n} C_\nu y_\nu(x) + \int_a^b g(x,\xi) f(\xi) d\xi, \tag{31}$$

where $C_1,\ldots,C_n$ are constants, and the function

$$g(x,\xi) = \pm \frac{1}{2W(\xi)} \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1^{(n-2)}(\xi) & y_2^{(n-2)}(\xi) & & y_n^{(n-2)}(\xi) \\ \cdot & \cdot & \cdots & \cdot \\ y_1(\xi) & y_2(\xi) & \cdots & y_n(\xi) \end{vmatrix} \tag{32}$$

the positive sign being taken if $x > \xi$, and the negative sign if $x < \xi$.

We next demand that the function y in (31) shall satisfy the boundary conditions $U_\nu(y) = 0, \nu = 1,2,\ldots,n$; i.e. that it shall be a solution of the boundary-value problem

$$l(y) = \lambda y + f, \qquad U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n.$$

This requires that

$$\sum_{j=1}^{n} C_j U_\nu(y_j) + \int_a^b f(\xi) U_\nu(g) d\xi = 0.$$

By solving these equations for C_j and substitution in (31), we get

$$y(x) = \int_a^b G(x,\xi,\lambda) f(\xi) d\xi \tag{33}$$

with

$$G(x,\xi,\lambda) = \frac{(-1)^n}{\Delta(\lambda)} H(x,\xi,\lambda), \tag{34}$$

where

$$\Delta(\lambda) = \begin{vmatrix} U_1(y_1) & U_1(y_2) & \ldots & U_1(y_n) \\ U_2(y_1) & U_2(y_2) & \ldots & U_2(y_n) \\ \cdot & & & \\ U_n(y_1) & U_n(y_2) & & U_n(y_n) \end{vmatrix} \tag{35}$$

and

$$H(x,\xi,\lambda) = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) & g(x,\xi) \\ U_1(y_1) & U_1(y_2) & \ldots & U_1(y_n) & U_1(g) \\ U_2(y_1) & U_2(y_2) & \ldots & U_2(y_n) & U_2(g) \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ U_n(y_1) & U_n(y_2) & & U_n(y_n) & U_n(g) \end{vmatrix} \tag{36}$$

If λ is not an eigenvalue of the operator L, then $\Delta(\lambda) \neq 0$, and the formulae (33) and (34) are meaningful. Formula (33) shows that $G(x,\xi,\lambda)$ is the Green's function of the operator $L - \lambda 1$. Hence:

VII. *The Green's function* $G(x,\xi,\lambda)$ *of the operator* $L - \lambda 1$ *is determined by the formulae* (32), (34), (35) *and* (36).

8. The Analytic Nature of the Green's Function of the Operator $L - \lambda 1$

The functions $\Delta(\lambda)$ and $H(x,\xi,\lambda)$, as may be seen immediately from (35) and (36), are integral, analytic functions of the parameter λ. So (34) implies:

VIII. *Green's function* $G(x,\xi,\lambda)$ *for the operator* $L - \lambda 1$ *is a meromorphic function of the parameter* λ, *and its poles can only be the eigenvalues of the operator.*

Let λ_0 be a simple zero of the function $\Delta(\lambda)$. Then λ_0 can also only be a simple pole of the function $G(x,\xi,\lambda)$, so that

$$G(x,\xi,\lambda) = \frac{R(x,\xi)}{\lambda - \lambda_0} + G_1(x,\xi,\lambda), \tag{37}$$

where $G_1(x,\xi,\lambda)$ is regular in a neighbourhood of the point λ_0.

If, then, λ_0 is not a singular point of the function $G(x,\xi,\lambda)$, we have to put $R(x,\xi) = 0$.

By a well-known formula in the theory of residues, we have

$$R(x,\xi) = (-1)^n \frac{H(x,\xi,\lambda_0)}{\Delta'(\lambda_0)} .$$

In the expansion of the determinant $H(x,\xi,\lambda)$ by elements of the first row, the coefficient of $g(x,\xi)$ is $\Delta(\lambda_0)$ and $= 0$; hence the function $R(x,\xi)$ and its first n derivatives with respect to x are continuous in the square $a \leqslant x,\xi \leqslant b$. Finally, it follows from (36) that $R(x,\xi)$ is a linear combination of the functions $y_1(x),\ldots,y_n(x)$; consequently, for a fixed $\xi, R(x,\xi)$ satisfies the equation

$$l(R) - \lambda_0(R) = 0.$$

We also derive from formula (36) that $H(x,\xi,\lambda)$, and therefore $R(x,\xi)$ also, satisfy the boundary conditions.

Therefore, for every fixed ξ, $R(x,\xi)$ is an eigenfunction of the operator L for the eigenvalue λ_0. Since, however, λ_0 is a simple zero of the function $\Delta(\lambda)$, there is only one eigenfunction $y_0(x)$ corresponding to it, up to a factor independent of x, for the operator L. We must therefore have

$$R(x,\xi) = a(\xi)y_0(x). \tag{38}$$

We now put $G^*(x,\xi,\lambda) = \overline{G(\xi,x,\lambda)}$; then $G^*(x,\xi,\lambda)$ is the Green's function for the operator $L^* - \bar{\lambda}1$, provided that the coefficients of the expression $l(y)$ satisfy the differentiability requirements needed for l^* to exist.

From equation (37), on the other hand, we infer that

$$G^*(x,\xi,\lambda) = \frac{\overline{R(\xi,x)}}{\bar{\lambda} - \bar{\lambda}_0} + \overline{G_1(\xi,x,\lambda)}.$$

So, for a fixed $\xi, \overline{R(\xi,x)}$ is an eigenfunction of the operator L^*, corresponding to the eigenvalue $\bar{\lambda}_0$. If we denote one of these functions by $z_0(x)$, we then have

$$\overline{R(\xi,x)} = b(\xi)z_0(x).$$

Hence

$$R(x,\xi) = \overline{b(x).z_0(\xi)}.$$

Comparing this formula with (38), we find

$$R(x,\xi) = cy_0(x)\overline{z_0(\xi)}. \tag{39}$$

It remains to determine the constant c.

From (37) and (39), we deduce that

$$(\lambda - \lambda_0) \int_a^b G(x,\xi,\lambda)y_0(\xi)d\xi = cy_0(x) \int_a^b y_0(\xi)\overline{z_0(\xi)}d\xi + (\lambda - \lambda_0) \int_a^b G_1(x,\xi,\lambda)y_0(\xi)d\xi. \tag{40}$$

The integrand in the last term is a regular function of the parameter λ in a neighbourhood about the point λ_0; hence we find from (40) that

$$\lim_{\lambda\to\lambda_0} (\lambda - \lambda_0) \int_a^b G(x,\xi,\lambda)y_0(\xi)d\xi = cy_0(x) \int_a^b y_0(\xi)\overline{z_0(\xi)}d\xi. \tag{41}$$

On the other hand,

$$(L - \lambda 1)y_0 = (\lambda_0 - \lambda)y_0,$$

whence

$$(L - \lambda 1)^{-1}y_0 = \frac{1}{\lambda_0 - \lambda} y_0.$$

Since $G(x,\xi,\lambda)$ is the Green's function for the operator $L - \lambda 1$, the last equation can be written in the form

$$\int_a^b G(x,\xi,\lambda)y_0(\xi)d\xi = \frac{1}{\lambda_0 - \lambda} y_0(x).$$

By substitution in (41) we obtain

$$-y_0(x) = cy_0(x) \int_a^b y_0(\xi)\overline{z_0(\xi)}d\xi,$$

so that c is given by

$$c = -\frac{1}{\int_a^b y_0(\xi)\overline{z_0(\xi)}d\xi},$$

and therefore

$$R(x,\xi) = -\frac{y_0(x)\overline{z_0(\xi)}}{\int_a^b y_0(\xi)\overline{z_0(\xi)}d\xi}.$$

It follows from (37) that:

IX. *For every simple zero* λ_0 *of the function* $\Delta(\lambda)$,

$$G(x,\xi,\lambda) = -\frac{y_0(x)\overline{z_0(\xi)}}{(\lambda - \lambda_0) \int_a^b y_0(\xi)\overline{z_0(\xi)}d\xi} + G_1(x,\xi,\lambda), \tag{42}$$

where $G_1(x,\xi,\lambda)$ *is regular in a neighbourhood of the point* λ_0.

The formula becomes particularly simple if the functions $y_0(x)$ and $z_0(x)$ are normalized so that

$$\int_a^b y_0(\xi)\overline{z_0(\xi)}d\xi \doteq 1.$$

It then becomes

$$G(x,\xi,\lambda) = -\frac{y_0(x)\overline{z_0(\xi)}}{\lambda - \lambda_0} + G_1(x,\xi,\lambda). \tag{43}$$

9. The Case of a Multiple Pole of the Green's Function

The above results may be generalized to the case of a multiple zero λ_0 of the function $\Delta(\lambda)$. In general, λ_0 is then also a multiple pole of the Green's function $G(x,\xi,\lambda)$. We shall again assume that $G(x,\xi,\lambda)$ is the Green's function for the boundary-value problem

$$l(y) = f, \qquad U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n, \tag{44}$$

where the coefficients of the differential expression $l(y)$ and of the forms $U_\nu(y)$ are integral, analytic functions of the parameter λ; the boundary-value problem $(L - \lambda 1)y = f$ is obviously a special case of this boundary-value problem. We observe that all the results of sections 7 and 8 carry over completely to the general case, $y_1(x,\lambda), y_2(x,\lambda), \ldots, y_n(x,\lambda)$ now being understood to be a fundamental system of solutions of the equation $l(y) = 0$, satisfying the same initial conditions as before, viz.

$$y_k^{(\nu-1)}(a,\lambda) = \begin{cases} 1 \text{ for } k = \nu, \\ 0 \text{ for } k \neq \nu. \end{cases}$$

Let, then, λ_0 now be a pole of the ν-th order for the Green's function $G(x,\xi,\lambda)$ for the boundary-value problem (44); then

$$G(x,\xi,\lambda) = \frac{R(x,\xi)}{(\lambda - \lambda_0)^\nu} + \frac{R_1(x,\xi)}{(\lambda - \lambda_0)^{\nu-1}} + \ldots + \frac{R_{\nu-1}(x,\xi)}{\lambda - \lambda_0} + G_1(x,\xi,\lambda), \tag{45}$$

$G_1(x,\xi,\lambda)$ being a function regular in a neighbourhood of the point λ_0.

For a fixed ξ, the function $G(x,\xi,\lambda)$ satisfies the equation $l(G) = 0$ in each of the intervals $[a,\xi)$ and $(\xi,b]$. By substituting for the function G its expression as in (45), we obtain the relations, for $\lambda = \lambda_0$:

$$\left.\begin{aligned} &l(R) && = 0, \\ &l(R_1) + \frac{1}{1!}\frac{\partial}{\partial\lambda}l(R) && = 0, \\ &\cdot \qquad \cdot \qquad \cdots \qquad \cdot && \cdot \\ &l(R_{\nu-1}) + \frac{1}{1!}\frac{\partial}{\partial\lambda}l(R_{\nu-2}) + \ldots + \frac{1}{(\nu-1)!}\frac{\partial^{\nu-1}}{\partial\lambda^{\nu-1}}l(R) && = 0. \end{aligned}\right\} \tag{46}$$

Moreover, it follows from the formulae (34) and (36) that, with a fixed ξ and for $\lambda = \lambda_0$, the functions $R(x,\xi), R_1(x,\xi), \ldots, R_{\nu-1}(x,\xi)$ satisfy the boundary conditions

$$\sum_{p=0}^{\mu} \frac{1}{p!} \frac{\partial^p}{\partial \lambda^p} U_q(R_{\mu-p}) = 0, \; R_0 = R, \; \mu = 0,1,\ldots,(\nu - 1), \; q = 1,2,\ldots,n. \tag{47}$$

By making use of the relations (46) and (47) as well as the definition of the eigenfunctions and the associated functions (§2.3), we can satisfy ourselves of the truth of the following theorem, which we merely state without proof (see [45a]):

X. *The principal part of the Green's function in a neighbourhood of the zero* λ_0 *of the function* $\Delta(\lambda)$ *has the form*

$$\sum_{j=1}^{p} \left\{ \frac{\overline{z_j(\xi)}y_j(x)}{(\lambda - \lambda_0)^{m_j}} + \frac{\overline{z_j(\xi)}y_{j1}(x) + \overline{z_{j1}(\xi)}y_j(x)}{(\lambda - \lambda_0)^{m_j - 1}} + \cdots \right.$$
$$\left. + \frac{\overline{z_j(\xi)}y_{j,m_j-1}(x) + \overline{z_{j1}(\xi)}y_{j,m_j-2}(x) + \cdots + \overline{z_{j,m_j-1}(\xi)}y_j(x)}{\lambda - \lambda_0} \right\} \tag{48}$$

where

$$y_j, y_{j1}, \ldots, y_{j,m_j-1}, \qquad j = 1,2,\ldots,p$$

is an arbitrary, canonical system of eigenfunctions and associated functions of the boundary-value problem concerned, while

$$z_j, z_{j1}, \ldots, z_{j,m_j-1}, \qquad j = 1,2,\ldots,p$$

is an appropriately normalized, canonical system of eigenfunctions and associated functions for the adjoint boundary-value problem.

CHAPTER II

ASYMPTOTIC BEHAVIOUR OF THE EIGENVALUES AND EIGENFUNCTIONS.

EXPANSION OF A PRESCRIBED FUNCTION IN TERMS OF EIGENFUNCTIONS OF A DIFFERENTIAL OPERATOR

§4. *The Asymptotic Behaviour of the Eigenvalues and Eigenfunctions for Large Values of* $|\lambda|$

1. Statement of the Problem

For large values of $|\lambda|$, approximation formulae and, indeed, asymptotic formulae can be given for the eigenvalues and eigenfunctions of a differential operator. Such formulae are not only of interest in themselves, but they also find application at a decisive point in the proof of certain theorems in the theory of differential operators, particularly in expansion theorems.

It turns out that the behaviour of the eigenvalues λ_ν and the eigenfunctions for an arbitrary differential operator as $|\lambda_\nu|$ increases can, to a first approximation, be characterized by the eigenvalues and eigenfunctions of the operator which is generated by the same boundary conditions but the simplest possible differential expression of the nth order, $l(y) = \frac{d^n y}{dx^n}$.

We shall first investigate the asymptotic behaviour of solutions of $l(y) = \lambda y$ for large $|\lambda|$, where $l(y)$ is an arbitrary differential expression. Then by substituting the asymptotic approximation expressions so obtained into the equation $\Delta = 0$ (cf. §2.1), we shall obtain the corresponding asymptotic estimates for the eigenvalues.

First we put $\lambda = -\rho^n$; the equation $l(y) = \lambda y$ then takes the form

$$l(y) + \rho^n y = 0, \tag{1}$$

or, in more detail,

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + p_n(x)y + \rho^n y = 0. \tag{2}$$

[For simplicity, we have at first taken $p_0(x) \equiv 1$; see §4.11(b) for the case

$p_0(x) \not\equiv 1$]. Without loss of generality, we may assume that $p_1(x) \equiv 0$. For if $p_1(x) \not\equiv 0$, then by using the substitution

$$y = \bar{y}.\exp\left(-\frac{1}{n}\int p_1(x)dx\right),$$

we find that equation (2) takes the form

$$\frac{d^n\bar{y}}{dx^n} + \bar{p}_2(x)\frac{d^{n-2}\bar{y}}{dx^{n-2}} + \ldots + \bar{p}_n(x)\bar{y} + \rho^n\bar{y} = 0,$$

where $\bar{p}_2(x),\ldots,\bar{p}_n(x)$ are likewise continuous with respect to x in the interval $[a,b]$, and ρ has not changed.

Also without loss of generality we can replace the interval $[a,b]$ by $[0,1]$, for we can return immediately to the general case by making the substitution

$$x = a + (b - a)t.$$

2. The Domains S and T

We divide the complex ρ-plane into $2n$ sectors S_k, $k = 0,1,\ldots,(2n - 1)$, defined by

$$\frac{k\pi}{n} \leqslant \arg\rho \leqslant \frac{(k+1)\pi}{n} \tag{3}$$

(Fig. 7).

As we shall see later, the asymptotic formulae for the solution y of equation (2) depend essentially on which sector S_k the point ρ lies in.

We denote by

$$\omega_1,\omega_2,\ldots,\omega_n$$

the different nth roots of -1 arranged in an order in each case to suit later requirements.

Later we shall need to use the following essential properties of the sectors S_k.

I. *For each of the sectors S_k the numbers $\omega_1,\omega_2,\ldots,\omega_n$ can be ordered in such a way that, for all $\rho \in S_k$, the inequalities*

$$\mathscr{R}(\rho\omega_1) \leqslant \mathscr{R}(\rho\omega_2) \leqslant \ldots \leqslant \mathscr{R}(\rho\omega_n) \tag{4}$$

hold, where $\mathscr{R}(z)$ means the real part of z.

Proof. It is sufficient to prove the assertion for two adjacent sectors S_k. For, multiplication by any particular nth root of unity, say

$$\varepsilon = e^{2l\pi i/n},$$

transforms each sector S_k into the sector $S_{k'}$ obtained from it by a rotation

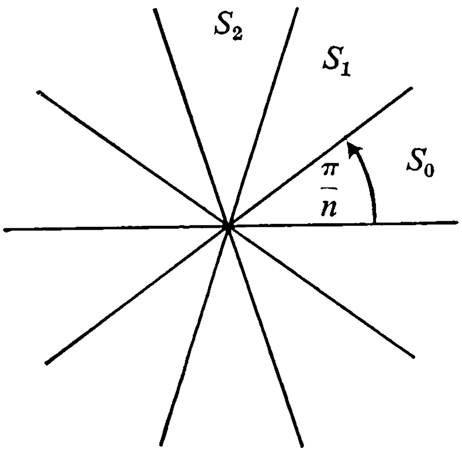

Fig. 7.

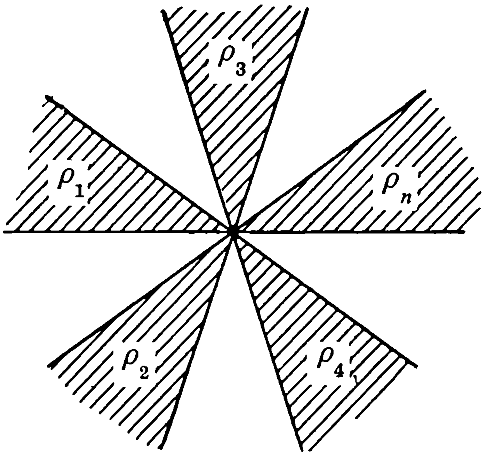

Fig. 8.

through the angle $\frac{2\pi l}{n}$, $0 \leqslant k \leqslant 2n-1$, and we have $k' \equiv k \,(\text{mod} 2)$; and the numbers

$$\varepsilon^{-1}\omega_1, \varepsilon^{-1}\omega_2, \ldots, \varepsilon^{-1}\omega_n$$

differ from the numbers $\omega_1, \omega_2, \ldots, \omega_n$ only in their order.

The inequalities (4) can now be written in the form

$$\mathscr{R}(\rho\varepsilon\,.\,\varepsilon^{-1}\omega_1) \leqslant \mathscr{R}(\rho\varepsilon\,.\,\varepsilon^{-1}\omega_2) \leqslant \ldots \leqslant \mathscr{R}(\rho\varepsilon\,.\,\varepsilon^{-1}\omega_n),$$

and by what has just been said, they provide the corresponding inequalities for the sector $S_{k'}$. Since l is arbitrary, they give the corresponding inequalities for all $S_{k'}$ with $k' - k \equiv 0$ (mod 2), provided only that they hold for the sector S_k. Hence it suffices to prove the assertion for two adjacent sectors, say S_{2n-1} and S_0.

Suppose n is odd: $n = 2\mu - 1$. Let, then, the order of the $\{\omega_k\}$ be chosen so that

$$\arg \omega_1 = \pi, \arg \omega_2 = \pi + \frac{2\pi}{n}, \arg \omega_3 = \pi - \frac{2\pi}{n}, \arg \omega_4 = \pi + \frac{4\pi}{n},$$

$$\arg \omega_5 = \pi - \frac{4\pi}{n}, \ldots, \arg \omega_n = \frac{\pi}{n}.$$

If we multiply any number $\rho \in S_{2n-1}$ in turn by $\omega_1, \omega_2, \ldots, \omega_n$, then the points $\rho_1, \rho_2, \ldots, \rho_n$ so obtained will be situated as shown in Fig. 8. For the domain S_{2n-1}, therefore, the inequalities (4) are geometrically obvious.

Similarly, we can show the validity of the assertion I for the domain S_0 and also for the case when n is even.

Later we shall find it expedient to consider more general domains, namely those which are obtained from the sectors S_k by a translation $\rho \to \rho - c$, where c is a fixed complex number. These new sectors with their vertices at the point $\rho = -c$ will correspondingly be denoted by T_k, $k = 0,1,2,\dots,2n-1$.

Taking account of the way in which the T_k are produced from the S_k by translation, we see that, for $\rho \in T_k$, the inequalities

$$\mathscr{R}((\rho + c)\omega_1) \leqslant \mathscr{R}((\rho + c)\omega_2) \leqslant \dots \leqslant \mathscr{R}((\rho + c)\omega_n) \tag{5}$$

hold, for a suitable ordering of the numbers $\omega_1,\omega_2,\dots,\omega_n$.

In the sequel we shall let ρ vary in a fixed domain T_k and so we shall write simply S and T instead of S_k and T_k. The order of the numbers $\omega_1,\omega_2,\dots,\omega_n$ will be such that for $\rho \in T$ the inequalities (5) are valid.

3. Reduction of the Equation $l(y) + \rho^n y = 0$ to an Integro-differential Equation

The derivation of the subsequent asymptotic approximation formulae depends on the fact that the equation

$$l(y) + \rho^n y = 0 \tag{6}$$

can be reduced to a certain equivalent integro-differential equation. To this end we put

$$m(y) = -p_2 y^{(n-2)} - \dots - p_n y, \tag{7}$$

and we can then write equation (6) in the form

$$y^{(n)} + \rho^n y = m(y). \tag{8}$$

The homogeneous, linear, differential equation $y^{(n)} + \rho^n y = 0$ has, for $\rho \neq 0$, the fundamental system

$$e^{\rho\omega_1 x}, e^{\rho\omega_2 x}, \dots, e^{\rho\omega_n x}.$$

If we regard (8) as an associated inhomogeneous equation with the inhomogeneous term $m(y)$, then the method of variation of the parameters gives the general solution y in the form

$$y = c_1 e^{\rho\omega_1 x} + \dots + c_n e^{\rho\omega_n x} + \int_0^x \frac{\omega_1 e^{\rho\omega_1(x-\xi)} + \dots + \omega_n e^{\rho\omega_n(x-\xi)}}{n\rho^{n-1}} m_\xi(y)\,d\xi, \tag{9}$$

where $m_\xi(y)$ denotes the value of $m(y)$ for $x = \xi$.

The equations (8) and (9) are equivalent in the sense that any solution of (9) with arbitrary constants $c_1,\dots,c_n$ is also a solution of (8), while conversely for each solution of (8) there exist constants $c_1,\dots,c_n$ with which (9) is satisfied. In this, the constants $c_1,\dots,c_n$ may depend on ρ. We keep k fixed at one of the values $k = 1,2,\dots,n$, and put

$$\left.\begin{aligned} c_j' &= c_j && \text{for } j = 1,2,\dots,k \\ c_j' &= c_j + \sum_{\alpha=k+1}^{n} \int_0^1 \frac{\omega_\alpha e^{-\rho\omega_\alpha\xi}}{n\rho^{n-1}} m_\xi(y)\,d\xi && \text{for } j = (k+1),\dots,n \end{aligned}\right\}. \tag{10}$$

Equation (9) then takes the form

$$y = c_1' e^{\rho\omega_1 x} + \dots + c_n' e^{\rho\omega_n x} + \frac{1}{n\rho^{n-1}} \int_0^x K_1(x,\xi,\rho) m_\xi(y) d\xi$$

$$- \frac{1}{n\rho^{n-1}} \int_x^1 K_2(x,\xi,\rho) m_\xi(y) d\xi, \tag{11}$$

where

$$K_1(x,\xi,\rho) = \sum_{\alpha=1}^{k} \omega_\alpha e^{\rho\omega_\alpha(x-\xi)}, \quad K_2(x,\xi,\rho) = \sum_{\alpha=k+1}^{n} \omega_\alpha e^{\rho\omega_\alpha(x-\xi)}. \tag{12}$$

Equation (11) is the integro-differential equation referred to.

4. A Lemma concerning a System of Integral Equations

LEMMA 1

If the system of integral equations

$$y_i(x) = f_i(x) + \sum_{j=1}^{r} \int_a^b A_{ij}(x,\xi,\lambda) y_j(\xi) d\xi, \qquad i = 1,2,\dots,r, \tag{13}$$

satisfies the following hypotheses:

1. *the functions $f_i(x)$ are continuous in the internal $[a,b]$;*
2. *for any fixed ξ, the functions $A_{ij}(x,\xi,\lambda)$ are continuous in $a \leqslant x < \xi$ and $\xi < x \leqslant b$;*
3. *for any fixed x and ξ $(a \leqslant x,\xi \leqslant b)$, the $A_{ij}(x,\xi,\lambda)$ are regular analytic functions of the parameter λ, defined in a region of the complex λ-plane extending to infinity;*
4. *there exist constants R and C such that*

$$|A_{ij}(x,\xi,\lambda)| \leqslant \frac{C}{|\lambda|}$$

for $|\lambda| > R$ and $a \leqslant x,\xi \leqslant b$:

then, for a sufficiently large R_0, there exists, for $|\lambda| \geqslant R_0$, one and only one solution

$$y_i(x) = y_i(x,\lambda), \qquad i = 1,\dots,r,$$

such that the functions $y_i(x,\lambda)$ are regular analytic functions of λ for $|\lambda| \geqslant R$, and such that

$$y_i(x,\lambda) = f_i(x) + O\left(\frac{1}{\lambda}\right) \text{ for } \lambda \to \infty.^1 \tag{14}$$

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[1] The expression $O(1/\lambda^k)$ denotes, here and elsewhere, a function of the form $f(x,\lambda)/\lambda^k$, where $|f(x,\lambda)|$ for $a \leqslant x \leqslant b$ and for sufficiently large $|\lambda|$ always remains less than a constant.

Proof. If the system (13) has a solution $y_i(x), i = 1,2,\dots,r$, then using the method of successive approximations we obtain the equations

$$\begin{aligned} y_i(x) &= f_i(x) + \sum_{j=1}^{r} \int_a^b A_{ij}(x,\xi,\lambda) f_j(\xi) d\xi \\ &\qquad + \sum_{j_1,j_2=1}^{r} \int_a^b \int_a^b A_{ij_1}(x,\xi_1,\lambda) A_{j_1 j_2}(\xi_1,\xi_2,\lambda) y_{j_2}(\xi_2) d\xi_1 d\xi_2 \\ &= \dots\dots \\ &= f_i(x) + \sum_{j=1}^{r} \int_a^b A_{ij}(x,\xi,\lambda) f_j(\xi) d\xi + \dots \\ &\qquad + \sum_{j_1,\dots,j_n=1}^{r} \int_a^b \cdots \int_a^b A_{ij_1}(x,\xi_1,\lambda) \dots A_{j_{n-1}j_n}(\xi_{n-1},\xi_n,\lambda) f_{j_n}(\xi_n) d\xi_1 \dots d\xi_n \\ &+ \sum_{j_1,\dots,j_{n+1}=1}^{r} \int_a^b \cdots \int_a^b A_{ij_1}(x,\xi_1,\lambda) \dots A_{j_n j_{n+1}}(\xi_n,\xi_{n+1},\lambda) y_{j_{n+1}}(\xi_{n+1}) d\xi_1 \dots d\xi_{n+1}. \end{aligned}$$

We put $B = \max |y_i(x)|, \quad a \leqslant x \leqslant b, \quad i = 1,2,\dots,r$.

Then, for $|\lambda| \geqslant R$, the terms in the last row of this equation chain add up to an amount not greater than

$$\frac{1}{|\lambda|^{n+1}} [(b-a)Cr]^{n+1} B,$$

and therefore tend to zero as $n \to \infty$, provided $|\lambda| > R_0$, where

$$R_0 = \max \{R, (b-a)Cr\}.$$

If this condition is satisfied, the solution is expressible in the form of an infinite series, viz.

$$\begin{aligned} y_i(x) &= y_i(x,\lambda) \\ &= f_i(x) + \sum_{j=1}^{r} \int_a^b A_{ij}(x,\xi,\lambda) f_j(\xi) d\xi \\ &\qquad + \sum_{j_1,j_2=1}^{r} \int_a^b \int_a^b A_{ij_1}(x,\xi,\lambda) A_{j_1 j_2}(\xi_1,\xi_2,\lambda) f_{j_2}(\xi_2) d\xi_1 d\xi_2 + \dots . \end{aligned}$$

Conversely, it can easily be seen that this series converges uniformly for every case with $|\lambda| \geqslant R_0, a \leqslant x \leqslant b$, and represents a solution of the system (13). Hence follow all the assertions of the lemma, and in particular the formula (14) and the regularity of the functions $y_i(x,\lambda)$ for $|\lambda| \geqslant R_0$.

5. Asymptotic Estimates for Solutions of the Equation $l(y) + \rho^n y = 0$

First we need estimates for the kernels K_1 and K_2 in (11) and for their derivatives.

LEMMA 2

There exists a constant C such that for all $\rho \in T$ *the following inequalities hold:*

$$\frac{\partial^\nu}{\partial x^\nu} K_1(x,\xi,\lambda) \leqslant C|\rho|^\nu k|e^{\rho\omega_k(x-\xi)}| \text{ for } 0 \leqslant \xi \leqslant x \leqslant 1, \tag{15 a}$$

$$\frac{\partial^\nu}{\partial x^\nu} K_2(x,\xi,\lambda) \leqslant C|\rho|^\nu(n-k)|e^{\rho\omega_k(x-\xi)}| \text{ for } 0 \leqslant x \leqslant \xi \leqslant 1, \tag{15 b}$$

$$\nu = 0,1,2,3,\ldots.$$

Proof. We choose a constant C such that, for all $j,k = 1,\ldots,n$ and all x,ξ in the interval $[a,b]$,

$$|e^{c(\omega_j-\omega_k)(x-\xi)}| \leqslant C; \tag{16}$$

this is possible, since the left-hand side of (16) is a continuous function of the variables x and ξ. It follows from the inequalities (5) that, for $\rho \in T$ and for $\alpha \leqslant k$,

$$\mathscr{R}(\rho\omega_\alpha) \leqslant \mathscr{R}(\rho\omega_\alpha + (\rho + c)(\omega_k - \omega_\alpha)),$$

which, for $0 \leqslant \xi \leqslant x \leqslant 1$, implies

$$|e^{\rho\omega_\alpha(x-\xi)}| \leqslant |e^{[\rho\omega_\alpha+(\rho+c)(\omega_k-\omega_\alpha)](x-\xi)}| \leqslant C|e^{\rho\omega_k(x-\xi)}|.$$

Therefore (see (12))

$$\left|\frac{d^\nu}{dx^\nu} K_1(x,\xi,\rho)\right| = \left|\sum_{\alpha=1}^{k} \rho^\nu \omega_\alpha^{\nu+1} e^{\rho\omega_\alpha(x-\xi)}\right| \leqslant C\,k\,|\rho|^\nu|e^{\rho\omega_k(x-\xi)}|$$

which is the inequality (15a).

Inequality (15 b) may be proved similarly.

THEOREM 1

If the functions $p_2,\ldots,p_n$ *are continuous*[1] *in the interval* $[0,1]$, *then the equation*

$$y^{(n)} + p_2y^{(n-2)} + \cdots + p_{n-1}y' + p_ny + \rho^n y = 0$$

has, for each region T of the complex ρ*-plane, n linearly independent solutions* $y_1,\ldots,y_n$, *which are regular for* $\rho \in T$ *and for sufficiently large* $|\rho|$, *and which,*

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[1] In fact, as will be seen from the proof given, the conclusion of the theorem remains true if $p_2,\ldots,p_n$ are arbitrary functions *summable* in the interval $[a,b]$.

with their derivatives, can be expressed in the form

$$\left.\begin{aligned} y_k &= e^{\rho\omega_k x}\left[1 + O\left(\frac{1}{\rho}\right)\right], \\ \frac{dy_k}{dx} &= \rho e^{\rho\omega_k x}\left[\omega_k + O\left(\frac{1}{\rho}\right)\right], \\ &\quad \cdot \qquad \cdot \\ \frac{d^{n-1}y_k}{dx^{n-1}} &= \rho^{n-1}e^{\rho\omega_k x}\left[\omega_k^{n-1} + O\left(\frac{1}{\rho}\right)\right]. \end{aligned}\right\} \tag{17}$$

Proof. Suppose that the equation (6) has a solution y_k for which

$$c'_\nu = 0 \text{ for } \nu \neq k \text{ and } c'_k = 1.$$

Then

$$y_k = e^{\rho\omega_k x} + \frac{1}{n\rho^{n-1}}\int_0^x K_1(x,\xi,\rho)m_\xi(y_k)d\xi - \frac{1}{n\rho^{n-1}}\int_x^1 K_2(x,\xi,\rho)m_\xi(y_k)d\xi. \tag{18 a}$$

The results of differentiating this equation $(n - 1)$ times, together with the original equation, give the system of equations

$$\begin{aligned} \frac{d^\nu y_k}{dx^\nu} &= \rho^\nu\omega_k^\nu e^{\rho\omega_k x} + \frac{1}{n\rho^{\nu-1}}\int_0^x \frac{\partial^\nu K_1(x,\xi,\rho)}{\partial x^\nu}\, m_\xi(y_k)d\xi \\ &\quad - \frac{1}{n\rho^{\nu-1}}\int_x^1 \frac{\partial^\nu K_2(x,\xi,\rho)}{\partial x^\nu}\, m_\xi(y_k)d\xi, \quad \nu = 0,1,2,\ldots,(n-1); \end{aligned} \tag{18 b}$$

for, the result of differentiating with respect to the upper and lower limit x is equal to zero when the sum of the two expressions is formed; this can be seen by differentiating the integral in (9) with respect to the upper limit, and using the known result $\omega_1^\nu + \ldots + \omega_n^\nu = 0$ for $\nu = 1,2,\ldots$.

If in (18) we put

$$\frac{d^\nu y_k}{dx^\nu} = \rho^\nu e^{\rho\omega_k x} z_{k\nu}, \tag{19}$$

we obtain the system of equations for the functions $z_{k\nu} = z_{k\nu}(x,\rho)$:

$$\begin{aligned} z_{k\nu}(x,\rho) &= \omega_k^\nu - \frac{1}{n\rho}\int_0^x e^{-\rho\omega_k(x-\xi)}\rho^{-\nu}\,\frac{\partial^\nu K_1(x,\xi,\rho)}{\partial x^\nu}\Big[p_2(\xi)z_{k,n-2}(\xi,\rho) \\ &\quad + \frac{1}{\rho}p_3(\xi)z_{k,n-3}(\xi,\rho) + \cdots \frac{1}{\rho^{n-2}}\,p_n(\xi)z_{k,0}(\xi,\rho)\Big]d\xi \\ &\quad + \frac{1}{n\rho}\int_x^1 e^{-\rho\omega_k(x-\xi)}\rho^{-\nu}\,\frac{\partial^\nu K_2(x,\xi,\rho)}{\partial x^\nu}\Big[p_2(\xi)z_{k,n-2}(\xi,\rho) \\ &\quad + \frac{1}{\rho}p_3(\xi)z_{k,n-3}(\xi,\rho) + \cdots + \frac{1}{\rho^{n-2}}\,p_n(\xi)z_{k,0}(\xi,\rho)\Big]d\xi \end{aligned} \tag{20}$$

If we now put

$$K_{k\nu\alpha}(x,\xi,\rho) = \begin{cases} -\frac{1}{n} e^{-\rho\omega_k(x-\xi)}\rho^{-\nu-\alpha+2} \frac{\partial^\nu K_1(x,\xi,\rho)}{\partial x^\nu} p_\alpha(\xi) \text{ for } \xi < x, \\ +\frac{1}{n} e^{-\rho\omega_k(x-\xi)}\rho^{-\nu-\alpha+2} \frac{\partial^\nu K_2(x,\xi,\rho)}{\partial x^\nu} p_\alpha(\xi) \text{ for } \xi > x; \end{cases}$$

$$k = 1,2,\ldots,n; \qquad \nu = 0,1,2,\ldots,(n-1); \qquad \alpha = 2,3,\ldots,n;$$

then the system (20) can be reduced to the form

$$z_{k\nu}(x,\rho) = \omega_k^\nu + \frac{1}{\rho}\sum_{\alpha=2}^{n}\int_0^1 K_{k\nu\alpha}(x,\xi,\rho)z_{k\alpha}(\xi,\rho)d\xi. \tag{21}$$

For a fixed k, the n equations obtained from (21) by putting $\nu = 0,1,\ldots,(n-1)$ form a system of integral equations for the functions $z_{k\nu}(x.\rho)$, $\nu = 0,1,\ldots,n$. It follows from Lemma 2 that all the functions $K_{k\nu\alpha}(x,\xi,\rho)$ are bounded for $0 \leqslant x, \xi \leqslant 1$, $\rho \in T$, and sufficiently large $|\rho|$. Hence all the assumptions in Lemma 1 of section 4 are satisfied. By this lemma, the system (21) has one and only one solution $z_{k\nu} = z_{k\nu}(x,\rho)$, which is analytic in ρ and which satisfies

$$z_{k\nu}(x,\rho) = \omega_k^\nu + O\left(\frac{1}{\rho}\right). \tag{22}$$

From this and from (19), the relations (17) follow immediately, and hence also the linear independence of the functions $y_k(x,\lambda)$, $k = 1,2,\ldots,n$.

It remains to prove that a solution $y_k(x,\lambda)$ of equation (6) exists, satisfying equation (18 a); to do this, it is only necessary to show that for arbitrarily chosen constants c'_ν (i.e. for numbers independent of ρ) there is a solution of equation (6) which for these c'_ν satisfies equation (11).

The equations (10) represent a linear transformation of the c_j into the c'_j (we remark that y and therefore also the expression $m_\xi(y)$ on the right-hand side of (10) depend linearly on $c_1,\ldots,c_n$). Clearly it is sufficient to show that the determinant of the transformation (10) is different from zero for sufficiently large $|\rho|$, $\rho \in T$; we can then solve equations (10) with arbitrarily prescribed c'_j for c_j. The solution y of equation (6), or, what is the same thing, of equation (9) for these values of c_j will then be the required solution.

If now the determinant of the transformation (10) vanishes for some $\rho \in T$, then the equations (10) with $c'_1 = c'_2 = \ldots = c'_n = 0$ for this ρ have a nontrivial solution c_j. The corresponding function y will consequently be a nontrivial solution of the equation

$$y = \frac{1}{n\rho^{n-1}}\int_0^x K_1(x,\xi,\rho)m_\xi(y)d\xi - \frac{1}{n\rho^{n-1}}\int_x^1 K_2(x,\xi,\rho)m_\xi(y)d\xi, \tag{23}$$

which is obtained from (11) on putting $c'_1 = c'_2 = \ldots = c'_n = 0$.

We shall show that, for sufficiently large $|\rho|$, this is impossible.

By differentiating (23) $(n-1)$ times and using the substitutions

$$z_\nu = \frac{d^\nu y}{dx^\nu}(\rho^\nu e^{\rho\omega_k x})^{-1}, \qquad \nu = 0,1,\dots,(n-1), \tag{24}$$

we obtain for the functions z_ν the system of equations

$$z_\nu(x,\rho) = \tag{25}$$

$$-\frac{1}{n\rho^{n-1}}\int_0^x \rho^{-\nu}e^{-\rho\omega_k(x-k)}\frac{\partial^\nu K_1(x,\xi,\rho)}{\partial x^\nu}\{\rho^{n-2}p_2(\xi)z_{n-2}(\xi,\rho)+\dots+p_n(\xi)z_0(\xi,\rho)\}d\xi +$$

$$\frac{1}{n\rho^{n-1}}\int_x^1 \rho^{-\nu}e^{-\rho\omega_k(x-k)}\,\frac{\partial^\nu K_2(x,\xi,\rho)}{\partial x^\nu}\,\{\rho^{n-2}p_2(\xi)z_{n-2}(\xi,\rho)+\dots+p_n(\xi)z_0(\xi,\rho)\}d\xi.$$

Let

$$m(\rho) = \max_{0\leqq x\leqq 1}|z_\nu(x,\rho)|, \qquad \nu = 0,1,\dots,(n-1).$$

By using Lemma 2 to estimate the right-hand side of (25), we obtain the inequality

$$|z_\nu(x,\rho)| \leqslant \Bigg[\frac{1}{n|\rho|}\,Ck\int_0^x\left\{|p_2|+\dots+\frac{|p_n|}{|\rho|^{n-2}}\right\}d\xi$$

$$+\frac{1}{n|\rho|}\,C(n-k)\int_x^1\left\{|p_2|+\dots+\frac{|p_n|}{|\rho|^{n-2}}\right\}d\xi\Bigg]m(\rho).$$

Since the left-hand side attains its maximum $m(\rho)$, it follows that

$$m(\rho) \leqslant m(\rho)\frac{C}{|\rho|}\int_0^1\left\{|p_2|+\dots+\frac{|p_n|}{|\rho|^{n-2}}\right\}d\xi \leqslant m(\rho)\frac{C_1}{|\rho|},$$

where C_1 is a certain constant.

For large $|\rho|$ this inequality is possible only if $m(\rho)=0$; hence certainly we have $z_\nu(x,\rho)\equiv 0$. It follows from this immediately by (24), for $\nu = 0$, that $y \equiv 0$. This proves the theorem completely.

Remark. The equation

$$U^{(n)} + \rho^n U = 0$$

has, for $\rho \neq 0$, the fundamental system of solutions

$$U_k = e^{\rho\omega_k x}, \qquad k = 1,2,\dots,n.$$

From formula (17) we deduce that the solutions y_k in Theorem 1 can be approximated by these functions U_k for large values of $|\rho|$.

In other words, for large values of $|\rho|$, only the first and last terms in the equation

$$y^{(n)} + p_2 y^{(n-2)} + \dots + p_n y + \rho^n y = 0$$

play an essential part.

In the case of a second-order equation of the form $y'' + p(x)y - \rho^2 y = 0$, there are four of the domains S (fig. 9); by the theorem just proved there exist in each of them linearly independent solutions y_1 and y_2 which can be represented asymptotically in the form

$$y_1 = e^{i\rho x}\left[1 + O\left(\frac{1}{\rho}\right)\right],$$

$$y_2 = e^{-i\rho x}\left[1 + O\left(\frac{1}{\rho}\right)\right].$$

For real and positive ρ it is often convenient to replace these solutions by the linear combinations of them

$$\frac{y_1 + y_2}{2} = \cos(\rho x) + O\left(\frac{1}{\rho}\right),$$

$$\frac{y_1 - y_2}{2i} = \sin(\rho x) + O\left(\frac{1}{\rho}\right).$$

The asymptotic behaviour of solutions of differential equations for large values of the parameter has been investigated in many works. Additional information and references can be found in the books [17], [97], [95b].

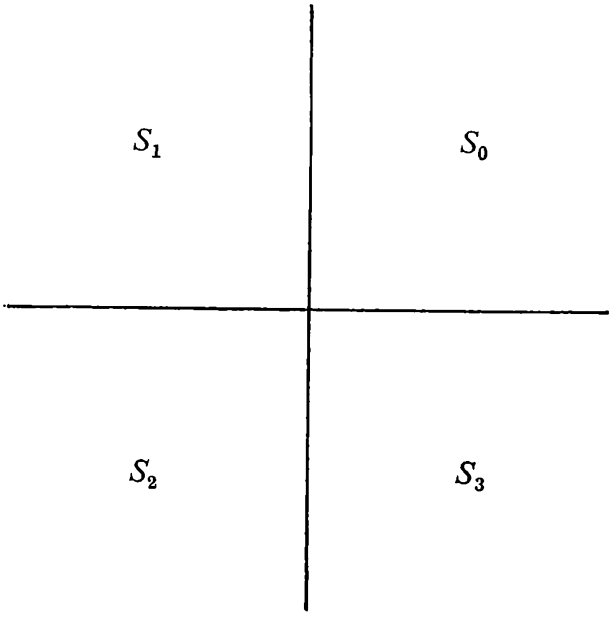

Fig. 9.

6. Refinement of the Asymptotic Formulae

If the coefficients $p_2,\ldots,p_n$ of the differential expression are not only themselves continuous but also have continuous derivatives up to a certain order m, then the asymptotic formulae already obtained can be made more precise.

Suppose, for example, that the coefficient $p_2(\xi)$ has a continuous first derivative $p_2'(\xi)$.

We substitute in the right-hand side of (18 a), instead of the functions y_k and their derivatives, the expressions from (17). Then using the estimates (15 a) and (15 b) for $\nu = 0$, we obtain:

$$y_k = e^{\rho\omega_k x} - \frac{1}{n\rho}\int_0^x K_1(x,\xi,\rho)e^{\rho\omega_k\xi}\omega_k^{n-2}p_2(\xi)d\xi$$

$$+ \frac{1}{n\rho}\int_x^1 K_2(x,\xi,\rho)e^{\rho\omega_k\xi}\omega_k^{n-2}p_2(\xi)d\xi + \mathrm{O}\left(\frac{1}{\rho^2}\right)e^{\rho\omega_k x} \quad (26)$$

But by (12)

$$\int_0^x K_1(x,\xi,\rho)e^{\rho\omega_k\xi}p_2(\xi)d\xi = e^{\rho\omega_k x}\left(\sum_{\alpha=1}^{k-1}\omega_\alpha\int_0^x e^{-\rho(\omega_k-\omega_\alpha)(x-\xi)}p_2(\xi)d\xi\right.$$

$$\left.+\ \omega_k\int_0^k p_2(\xi)d\xi\right),$$

and by integrating by parts and using the inequalities (5) we can easily convince ourselves that the first addend in the brackets is an expression of the form $O\left(\frac{1}{\rho}\right)$. Hence

$$\int_0^x K_1(x,\xi,\rho)e^{\rho\omega_k\xi}p_2(\xi)d\xi = e^{\rho\omega_k x}\left[\omega_k\int_0^x p_2(\xi)d\xi + \mathrm{O}\left(\frac{1}{\rho}\right)\right],$$

and similarly

$$\int_x^1 K_2(x,\xi,\rho)e^{\rho\omega_k\xi}p_2(\xi)d\xi = e^{\rho\omega_k x}\mathrm{O}\left(\frac{1}{\rho}\right).$$

Substituting these expressions in (26), we get

$$y_k = e^{\rho\omega_k}\left[1 + \frac{y_{k0}(x)}{\rho} + \mathrm{O}\left(\frac{1}{\rho^2}\right)\right], \quad (27)$$

where

$$y_{k0}(x) = -\frac{\omega_k}{n}\int_0^x p_2(\xi)d\xi. \quad (28)$$

Similarly, by substituting the expressions from (17) into the right-hand side of (18 b), we obtain:

$$\frac{d^\nu y_k}{dx^\nu} = \rho^\nu \omega_k^\nu e^{\rho\omega_k x}\left[1 + \frac{y_{k\nu}(x)}{\rho} + O\left(\frac{1}{\rho^2}\right)\right], \quad \nu = 0,1,\ldots,(n-1); \tag{29}$$

in this case it can readily be checked that the first two terms in the brackets in formula (29) result from the fact that we differentiate the expression

$$e^{\rho\omega_k x}\left[1 + \frac{y_{k0}(x)}{\rho}\right]$$

in (27) ν times and, after taking out the factor $\rho^\nu \omega_k^\nu e^{\rho\omega_k x}$, discard all terms of order $O\left(\frac{1}{\rho^2}\right)$.

If p_2 and p_3 have continuous derivatives up to the second and first order respectively, then by substituting the expressions (29) in the right-hand sides of (18 a) and (18 b), we derive in a similar way to the above

$$\frac{d^\nu y_k}{dx^\nu} = \rho^\nu \omega_k^\nu e^{\rho\omega_k x}\left[1 + \frac{y_{k\nu 1}}{\rho} + \frac{y_{k\nu 2}}{\rho^2} + O\left(\frac{1}{\rho^3}\right)\right]. \tag{30}$$

Repetition of the argument leads to the following

Corollary. If the functions $p_2, p_3, \ldots, p_n$ *have continuous derivatives up to the orders* $m, (m-1), (m-2), \ldots$ *respectively in the interval* $[0,1]$, *then the following asymptotic formulae hold for the solutions* $y_1, y_2, \ldots, y_n$ *constructed in Theorem* 1:

$$y_k = e^{\rho\omega_k x}\left[1 + \frac{y_{k01}(x)}{\rho} + \frac{y_{k02}(x)}{\rho^2} + \ldots + \frac{y_{k0m}(x)}{\rho^m} + O\left(\frac{1}{\rho^{m+1}}\right)\right], \tag{31 a}$$

$$\frac{\partial^\nu y_k}{\partial x^\nu} = \rho^\nu \omega_k^\nu e^{\rho\omega_k x}\left[1 + \frac{y_{k\nu 1}(x)}{\rho} + \frac{y_{k\nu 2}(x)}{\rho^2} + \ldots + \frac{y_{k\nu m}(x)}{\rho^m} + O\left(\frac{1}{\rho^{m+1}}\right)\right] \tag{31 b}$$

$$k = 1,2,\ldots,n; \qquad \nu = 1,2,\ldots,n-1,$$

where $y_{k\nu 1}(x), \ldots, y_{k\nu m}(x)$ are continuous functions in the interval $[0,1]$.

Remark 1. *The formula* (31 b) *follow from* (31 a) *by termwise differentiation, taking the factor* $\rho^\nu a_k^\nu e^{\rho\omega_k x}$ *outside the brackets, and collecting all those terms in the brackets which contain factors* $1/\rho^k$ *with* $k > m$ *into an expression of the form* $O\left(\frac{1}{\rho^{m+1}}\right)$.

We can easily satisfy ourselves that this is so, by considering that the term $O\left(\frac{1}{\rho^{m+1}}\right)$ in (31 a) is an expression of the form

$$\frac{1}{\rho^{m+1}}\left[\int_0^x e^{-\rho\omega_k(x-\xi)}K_1(x,\xi,\rho)\varphi(\xi,\rho)d\xi - \int_x^1 e^{-\rho\omega_k(x-\xi)}K_2(x,\xi,\rho)\varphi(\xi,\rho)d\xi\right],$$

where $\varphi(\xi,\rho)$ denotes a certain bounded function; from the Lemma in section 5 it then follows that the νth derivative of this expression with respect to x ($\nu = 1,2,\ldots,(n-1)$) is of the form $O\left(\frac{1}{\rho^{m+1-\nu}}\right)$.

The functions $y_{k\nu j}(x)$ can therefore be expressed in terms of the functions $y_{k0\mu}(x)$.

Remark 2. We can determine the functions $y_{k0\mu}(x)$, up to additive constants, even more simply by substituting the expressions (31 a) and (31 b) into the equation $l(y) + \rho^n y = 0$, and, after removing a factor $\rho^n e^{\rho\omega_k x}$, comparing terms with the same powers of ρ, from $1/\rho$ to $1/\rho^m$ inclusive. In this way we obtain a system of recursive differential equations of the first order, from which we can successively determine the functions $y_{k0\mu}(x)$, up to additive constants.

In fact, after the substitutions mentioned, and the removal of the factor, we obtain the result

$$\frac{A_1}{\rho} + \frac{A_2}{\rho^2} + \ldots + \frac{A_m}{\rho^m} + O\left(\frac{1}{\rho^{m+1}}\right) = 0, \tag{32}$$

where A_ν/ρ^ν denotes the sum of all terms of the equation which contain $1/\rho^\nu$ but not $1/\rho^{\nu+1}$.

Obviously, equation (32) can hold only if

$$A_1 = 0, \qquad A_2 = 0,\ldots, \qquad A_m = 0. \tag{33}$$

If we actually calculate the expressions for $A_1,A_2,\ldots,A_m$, we obtain from the equations (33) the following expressions for the functions $y_{k0\nu}$:

$$y_{k01} = \alpha_1 - \frac{1}{n\omega_k}\int_0^x p_2(\xi)d\xi, \tag{34 a}$$

$$y_{k0\nu} = \alpha_\nu - \frac{1}{n}\sum_{\alpha=0}^{\nu-1}\sum_{\beta=0}^{\nu-\alpha-1}\omega_k^{\alpha-\nu}\int_0^x p_{\nu+1-\alpha-\beta}(\xi)y_{k0\alpha}^{(\beta)}(\xi)d\xi, \quad \nu = 2,3,4,\ldots,m. \tag{34 b}$$

The constants α_ν ($\nu = 1,2,3,\ldots$) could already be found from the equations (18). Thus we saw earlier that $\alpha_1 = 0$, for example. In the same way the remaining constants α_ν can be determined.

7. Normalization of the Boundary Conditions

We wish to investigate the different systems $U_\nu(y),\nu = 1,2,\ldots,n$, of linear forms which define a given differential operator. If $y_0^{(k)}$ or $y_1^{(k)}$ appear explicitly in the form $U(y)$ but $y_0^{(\nu)}$ and $y_1^{(\nu)}$ do not, for any $\nu > k$, then we say that the form

$U(y)$ has *order* k.[1] We consider the forms $U_\nu(y)$ of order $(n - 1)$, if there are any. By replacing them, if necessary, by equivalent linear combinations, we can arrange that the maximum number of forms of order $(n - 1)$ is $\leqslant 2$. The remaining forms have orders $\leqslant (n - 2)$; we apply the same process to the forms of order $(n - 2)$ and reduce their number to a minimum; and so on.

The operations described are referred to as the *normalization of the boundary conditions*, and the boundary conditions finally resulting are said to be *normalized*. From the way in which they are constructed it follows that the *normalized boundary conditions must have the form*

$$U_\nu(y) \equiv U_{\nu 0}(y) + U_{\nu 1}(y) = 0, \tag{35}$$

where

$$U_{\nu 0}(y) = \alpha_\nu y_0^{(k_\nu)} + \sum_{j=0}^{k_\nu - 1} \alpha_{\nu j} y_0^{(j)}, \tag{36}$$

$$U_{\nu 1}(y) = \beta_\nu y_1^{(k_\nu)} + \sum_{j=0}^{k_\nu - 1} \beta_{\nu j} y_1^{(j)} \tag{37}$$

$$n - 1 \geqslant k_1 \geqslant k_2 \geqslant \ldots \geqslant k_n \geqslant 0, \qquad k_{\nu+2} < k_\nu,$$

and for each value of the suffix ν at least one of the numbers α_ν, β_ν is non-zero.

8. Regular Boundary Conditions

We consider a fixed domain S_k; as before, we number $\omega_1, \omega_2, \ldots, \omega_n$ so that, for $\rho \in S_k$,

$$\mathscr{R}(\rho\omega_1) \leqslant \mathscr{R}(\rho\omega_2) \leqslant \ldots \leqslant \mathscr{R}(\rho\omega_n).$$

For our later purposes it is convenient to particularize a class of boundary conditions which we shall refer to as *regular*. This class is defined differently for odd and even values of n, as follows:

(a) *n is odd*: $n = 2\mu - 1$.

The normalized boundary conditions (35) *are said to be regular if the numbers θ_0 and θ_1 defined by the identity*

$$\theta_0 + \theta_1 s = \begin{vmatrix} \alpha_1 \omega_1^{k_1} & & \alpha_1 \omega_{\mu-1}^{k_1} & (\alpha_1 + s\beta_1)\omega_\mu^{k_1} & \beta_1 \omega_{\mu+1}^{k_1} & & \beta_1 \omega_n^{k_1} \\ \alpha_2 \omega_1^{k_2} & & \alpha_2 \omega_{\mu-1}^{k_2} & (\alpha_2 + s\beta_2)\omega_\mu^{k_2} & \beta_2 \omega_{\mu+1}^{k_2} & & \beta_2 \omega_n^{k_2} \\ \cdot & \cdots & \cdot & & \cdot & \cdots & \cdot \\ \alpha_n \omega_1^{k_n} & \cdots & \alpha_n \omega_{\mu-1}^{k_n} & (\alpha_n + s\beta_n)\omega_\mu^{k_n} & \beta_n \omega_{\mu+1}^{k_n} & & \beta_n \omega_n^{k_n} \end{vmatrix} \tag{38}$$

are both different from zero.

---

[1] y_0, y_1 correspond to y_a, y_b of §1.3.

(b) *n is even: $n = 2\mu$.*

The normalized boundary conditions (35) *are said to be regular if the numbers θ_{-1} and θ_1 defined by the identity*

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s =$$

$$\begin{vmatrix} \alpha_1\omega_1^{k_1} & \cdots & \alpha_1\omega_{\mu-1}^{k_1} & (\alpha_1+s\beta_1)\omega_\mu^{k_1} & \left(\alpha_1+\dfrac{1}{s}\beta_1\right)\omega_{\mu+1}^{k_1} & \beta_1\omega_{\mu+2}^{k_1} & \cdots & \beta_1\omega_n^{k_1} \\ \alpha_2\omega_1^{k_2} & \cdots & \alpha_2\omega_{\mu-1}^{k_2} & (\alpha_2+s\beta_2)\omega_\mu^{k_2} & \left(\alpha_2+\dfrac{1}{s}\beta_2\right)\omega_{\mu+1}^{k_2} & \beta_2\omega_{\mu+2}^{k_2} & \cdots & \beta_2\omega_n^{k_2} \\ \cdot & \cdots & \cdot & \cdot & & & \cdots & \\ \alpha_n\omega_1^{k_n} & \cdots & \alpha_n\omega_{\mu-1}^{k_n} & (\alpha_n+s\beta_n)\omega_\mu^{k_n} & \left(\alpha_n+\dfrac{1}{s}\beta_n\right)\omega_{\mu+1}^{k_n} & \beta_n\omega_{\mu+2}^{k_n} & \cdots & \beta_n\omega_n^{k_n} \end{vmatrix} \qquad (39)$$

are different from zero.

This definition of regularity is *independent of the choice of the region S for which the numbers ω_k were arranged in order.* In order to show this, we first go back to section 2 and recall that there we were able to obtain the different ordered-arrangements of the ω_k belonging to individual domains S_k by multiplying by an nth root of unity the values of the ω_k in the order in which they occurred for one or other of two basic, adjacent domains, say the domains S_0 and S_{2n-1}. If now all the ω_k in equation (38) or (39) are multiplied by the same factor with modulus equal to 1, then the numbers θ will change by a common factor with modulus equal to 1. In particular, the ratios $\theta_0 : \theta_1$ or $\theta_0 : \theta_1 : \theta_{-1}$, and so also the roots of the equations $\theta_1\xi - \theta_0 = 0$ or $\theta_1\xi^2 + \theta_0\xi + \theta_{-1} = 0$, remain invariant; we shall apply this result later.

So it now remains to find out how the numbers θ change when we change from an appropriate domain S_k with k even to a suitable chosen domain $S_{k'}$ with k' odd. We shall, in fact, find that this change, too, can be achieved for each separate number θ by multiplication by a suitably chosen number with modulus equal to 1 but that now, in general, all the numbers θ will not be multiplied by the same factor.

We discuss this first for the case when n is odd:

If $n = 2\mu - 1$, the numbers $\psi_k = e^{i\pi + 2k\pi i/n}$, $k = 0, \pm 1, \pm 2, \ldots, \pm(\mu - 1)$ obviously represent all the distinct nth roots of -1. We can correlate any particular order of the ω_k to the ψ_k simply by specifying the row of suffixes. For example, the row of suffixes in the proper order for the inequalities (4) to hold in the domain S_{2n-1} runs:

$$(0, +1, -1, +2, -2, \ldots, (\mu - 1), -(\mu - 1)),$$

while that for the domain S_0 is

$$(0, -1, +1, -2, +2, \ldots, (-\mu - 1), (\mu - 1)).$$

Now, we have

$$\theta_0 = \begin{vmatrix} \alpha_1\omega_1^{k_1} & \dots & \alpha_1\omega_\mu^{k_1} & \beta_1\omega_{\mu+1}^{k_1} & \dots & \beta_1\omega_n^{k_1} \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ \alpha_n\omega_1^{k_n} & \dots & \alpha_n\omega_\mu^{k_n} & \beta_n\omega_{\mu+1}^{k_n} & \dots & \beta_n\omega_n^{k_n} \end{vmatrix}$$

and the determinant for θ_1 has the same form except that in the μth column the α_k are replaced by β_k.

Let us suppose that θ_0 and θ_1 have been formed for the domain S_{2n-1}, and let the corresponding numbers for the domain S_0 be denoted by $\tilde{\theta}_0$ and $\tilde{\theta}_1$. It is clear that $\tilde{\theta}_0,\tilde{\theta}_1$ are obtained from θ_0,θ_1 by interchanging the numbers ω_k in the second and third columns, in the fourth and fifth columns, etc. For n odd, this reduces to an actual interchange of columns in the determinant for just one of the numbers θ_0 or θ_1. This will become clear by considering the following arrays:

(a_1) μ even, $n = 4\nu - 1$.

$$\begin{array}{c|ccccccccccc|} \theta_0 & \alpha & \alpha & \alpha\dots & \alpha & \alpha & \alpha & \beta\dots & \beta & \beta \\ & (0, & 1,-1,\dots, & & (\nu-1), & -(\nu-1), & \nu,-\nu,\dots, & & (2\nu-1), & -(2\nu-1)) \\ \theta_1 & \alpha & \alpha & \alpha\dots & \alpha & \alpha & \beta & \beta\dots & \beta & \beta \end{array}$$

(a_2) μ odd, $\mu = 2\nu + 1$, $n = 4\nu + 1$.

$$\begin{array}{c|cccccccc|} \theta_0 & \alpha & \alpha & \alpha\dots\alpha & \alpha & \beta & \beta & \dots\beta & \beta \\ & (0, & 1,-1,\dots, & \nu, & -\nu, & (\nu+1), & -(\nu+1),\dots, & 2\nu, & -2\nu) \\ \theta_1 & \alpha & \alpha & \alpha\dots\alpha & \beta & \beta & \beta & \dots\beta & \beta \end{array}.$$

These arrays merely show, for each column of the determinants for θ_0 and θ_1, the ψ-suffix for the ω_k occurring in that column, and the letters α or β above and below this index shows whether one of the $\alpha_1,\dots,\alpha_n$ or one of the $\beta_1,\dots,\beta_n$ goes with it. For the transition from S_{2n-1} to S_0, the index row $(0,1,-1,\dots)$ is simply replaced by the row $(0,-1,1,\dots)$ and the (α,β)-distribution remains unaltered.

This can be accomplished in case (a_1) for θ_1, and in case (a_2) for θ_0, by a simple interchange of columns, and it follows that

$$\tilde{\theta}_1 = -\theta_1 \quad \text{for} \quad n = 4\nu - 1,$$

$$\tilde{\theta}_0 = \theta_0 \quad \text{for} \quad n = 4\nu + 1.$$

To calculate the value of the other θ in each of these cases, we proceed thus:

A multiplication by $\varepsilon = e^{2\pi i l/n}$ transforms the system $\omega_1,\ldots,\omega_n$ into itself; only the sequential order is permuted—and this simply by adding l to each number in the index row.

From the suffix row:

$$(\overset{\cdot}{0},\overset{\cdot}{1},\overset{\cdot}{-1},\overset{\cdots}{\ldots},\overset{\cdot}{(\nu-1)},\overset{\alpha}{-(\nu-1)},\overset{\alpha}{\nu},\overset{\beta}{-\nu},\overset{\beta}{(\nu+1)},\overset{\cdots}{\ldots},\overset{\cdot}{-(2\nu-1)})$$

the row:

$$(\overset{\cdot}{-1},\overset{\cdot}{0},\overset{\cdot}{-2},\overset{\cdot}{1},\overset{\cdot}{-3},\overset{\cdots}{\ldots},\overset{\cdot}{(\nu-2)},\overset{\alpha}{-\nu},\overset{\alpha}{(\nu-1)},\overset{\beta}{-(\nu+1)},\overset{\beta}{\nu},\overset{\cdots}{\ldots},\overset{\cdot}{(2\nu-2)},\overset{\cdot}{2\nu-1})$$

can be obtained (with $l = -1$).

We see that this last row can be brought into the form

$$(\overset{\cdot}{0},\overset{\cdot}{-1},\overset{\cdot}{+1},\overset{\cdot}{-2},\overset{\cdot}{+2},\overset{\cdots}{\ldots},\overset{\cdot}{-(\nu-1)},\overset{\alpha}{(\nu-1)},\overset{\alpha}{-\nu},\overset{\beta}{\nu},\overset{\beta}{-(\nu+1)},\overset{\cdot}{(\nu+1)},\ldots,$$
$$\overset{\cdot}{-(2\nu-1)},\overset{\cdot}{(2\nu-1)})$$

by interchanging the members of its first $(2\nu-1)$ pairs.

Hence it follows that

$$\tilde{\theta}_0 = -e^{2\pi i(k_1+\ldots+k_n)/n}\,\theta_0 \quad \text{for} \quad n = 4\nu - 1.$$

Similarly it can be shown that

$$\tilde{\theta}_1 = e^{-2\pi i(k_1+\ldots+k_n)/n}\,\theta_1 \quad \text{for} \quad n = 4\nu + 1.$$

These results imply that, for odd n, the definition of regularity is independent of the choice of the zone S_k. Moreover, it has been shown that:

The root of the equation $\theta_1\xi - \theta_0 = 0$ remains invariant for the transition from S_k to $S_{k'}$ if $k \equiv k' \pmod 2$, but for the transition from S_{2l+1} to S_{2l} it is multiplied by a factor $e^{\pm 2\pi i(k_1+\ldots+k_n)/n}$, where the plus sign is to be taken if $n = 4\nu + 1$, and the minus sign if $n = 4\nu - 1$. If $\xi^{(1)}$, $\xi^{(2)}$ are the roots for S_k with even and odd k, then $\xi^{(2)} = e^{\pm 2\pi i(k_1+\ldots+k_n)/n}\xi^{(1)}$.

For even n, $n = 2\mu$, we can attain our goal rather more conveniently. The numbers $\psi_k = e^{i\pi+k\pi i/n}$, $k = \pm 1,\pm 3,\pm 5,\ldots,\pm(n-1)$, now denote all the nth roots of -1, and so we can again characterize any special order of the numbers $\omega_1,\ldots,\omega_n$, and consequently any particular domain S_k, by giving the corresponding row of suffixes for its ψ_k. For the domains S_{2n-1}, S_0, S_1 respectively we obtain in this way the suffix rows:

$$(\ 1,-1,\ 3,-3, \qquad\qquad ,\ (n-1),-(n-1))$$
$$(-1,\ 1,-3,\ 3, \qquad\qquad ,-(n-1),\ (n-1))$$
$$(-1,-3,\ 1,-5,3,-7,5,\ldots,-(n-1),\ (n-3),\ (n-1)).$$

We again distinguish two cases:

(b$_1$) $$\mu = 2\nu,\ n = 4\nu.$$

The row of suffixes for S_1 can be obtained from that for S_0 by interchanging the second and third columns, the fourth and fifth, ..., the $(n-2)$th and $(n-1)$th.

If the corresponding columns are interchanged in the determinant (39) formed for the domain S_0, and if s is replaced by $1/s$, then clearly the determinant (39) for the domain S_1 is obtained.

Since the number of column interchanges carried out is odd, we have $D_1(s) = -D_0(1/s)$, where $D_k(s)$ denotes the determinant (39) for the domain S_k.

(b$_2$) $$\mu = 2\nu + 1,\ n = 4\nu + 2.$$

In this case we obtain in exactly the same way

$$D_{2n-1}(s) = -D_0\,(1/s).$$

From the equations obtained in cases (b$_1$) and (b$_2$) it follows, first of all, that the definition of regularity is independent of the special choice of the domain S_k.

We further notice that:

The roots ξ', ξ'' of the quadratic equation $D_k(s) = 0$ for the regular boundary conditions remain invariant when k is changed to k' with $(k' - k)$ even; but they go over into their reciprocals $1/\xi'$, $1/\xi''$ if $(k' - k)$ is odd.

Finally we remark that:

If the numbers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are all real, then the numbers θ go over into their complex conjugates for the transition from S_{2n-1} to S_0. In this case, then, the independence of the definition of regularity from choice of S_k follows immediately.

We shall now discuss a few examples of regular boundary conditions.

(a) *Conditions of Sturm Type for even n* $(n = 2\mu)$.

This is the name given to boundary conditions of the form:

$$\left.\begin{aligned} U_{j0}(y) &\equiv y_0^{(k_j)} + \sum_{\nu=1}^{k_j-1} \alpha_{j\nu} y_0^{(\nu)} = 0, \\ U_{j1}(y) &\equiv y_1^{(k'_j)} + \sum_{\nu=1}^{k'_j-1} \beta_{j\nu} y_0^{(\nu)} = 0, \end{aligned}\right\} j = 1,2,\ldots,\mu, \tag{40}$$

with $n - 1 \geqslant k_1 > k_2 > \ldots > k_\mu \geqslant 0$; $n - 1 \geqslant k'_1 > k'_2 > \ldots > k'_\mu \geqslant 0$.

Here half the boundary conditions contain only the function values of y and its derivatives at the point $x = 0$, and the other half of them only the values at the point $x = 1$.

In this case,

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \pm \begin{vmatrix} \omega_1^{k_1} & \dots & \omega_{\mu-1}^{k_1} & \omega_\mu^{k_1} & \omega_{\mu+1}^{k_1} & 0 & \dots & 0 \\ \cdot & \dots & \cdot & \cdot & \cdot & \cdot & \dots & \\ \omega_1^{k_\mu} & & \omega_{\mu-1}^{k_\mu} & \omega_\mu^{k_\mu} & \omega_\mu^{k_\mu} & 0 & \dots & 0 \\ 0 & & 0 & s\omega_\mu^{k'_1} & \frac{1}{s}\omega_{\mu+1}^{k'_1} & \omega_{\mu+2}^{k'_1} & \dots & \omega_n^{k'_1} \\ \cdot & \dots & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & & 0 & s\omega_\mu^{k'_\mu} & \frac{1}{s}\omega_{\mu+1}^{k'_\mu} & \omega_{\mu+2}^{k'_\mu} & \dots & \omega_n^{k'_\mu} \end{vmatrix}$$

(The ± sign occurs because the rows are not in the same order as in formula (39)). Consequently

$$\theta_0 = 0$$

$$\theta_1 = \pm \begin{vmatrix} \omega_1^{k_1} & \omega_{\mu-1}^{k_1} & \omega_{\mu+1}^{k_1} \\ \cdot & \cdot & \cdot \\ \omega_1^{k_\mu} & \omega_{\mu-1}^{k_\mu} & \omega_{\mu+1}^{k_\mu} \end{vmatrix} \begin{vmatrix} \omega_\mu^{k'_1} & \omega_{\mu+2}^{k'_1} & \dots & \omega_n^{k'_1} \\ \cdot & \cdot & \dots & \cdot \\ \omega_\mu^{k'_\mu} & \omega_{\mu+2}^{k'_\mu} & \dots & \omega_n^{k'_\mu} \end{vmatrix} \tag{41}$$

$$\theta_{-1} = \pm \begin{vmatrix} \omega_1^{k_1} & & \omega_\mu^{k_1} \\ \cdot & \dots & \cdot \\ \omega_1^{k_\mu} & & \omega_\mu^{k_\mu} \end{vmatrix} \begin{vmatrix} \omega_{\mu+1}^{k'_1} & & \omega_n^{k'_1} \\ \cdot & \dots & \cdot \\ \omega_{\mu+1}^{k'_\mu} & \dots & \omega_n^{k'_\mu} \end{vmatrix} \tag{42}$$

Hence *conditions of Sturm Type are regular provided that all the determinants in the formulae* (41) *and* (42) *do not vanish.*

If, for example,

$$k_2 = k_1 - 1,\ k_3 = k_1 - 2, \dots,\ k_\mu = k_1 - \mu + 1,$$

$$k'_2 = k'_1 - 1,\ k'_3 = k'_1 - 2, \dots,\ k'_\mu = k'_1 - \mu + 1,$$

these determinants are different from zero and so the conditions are regular.

(b) *Conditions of Periodic Type.*

This is the name given to conditions of the form:

$$U_\nu(y) \equiv y_0^{(\nu)} - y_1^{(\nu)} = 0, \qquad \nu = 0,1,2,\dots,(n-1).$$

Conditions of periodic type are regular.

For, if n is even ($n = 2\mu$), then

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \pm \begin{vmatrix} 1 & \dots & 1 & (1-s) & \left(1-\frac{1}{s}\right) & 1 & \dots & 1 \\ \omega_1 & \dots & \omega_{\mu-1} & (1-s)\omega_\mu & \left(1-\frac{1}{s}\right)\omega_{\mu+1} & \omega_{\mu+2} & \dots & \omega_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \omega_1^{n-1} & \dots & \omega_{\mu-1}^{n-1} & (1-s)\omega_\mu^{n-1} & \left(1-\frac{1}{s}\right)\omega_{\mu+2}^{n-1} & \omega_{\mu+2}^{n-1} & \dots & \omega_n^{n-1} \end{vmatrix}$$

$$= \pm C(1-s)\left(1-\frac{1}{s}\right),$$

where C is the Vandermonde determinant for the numbers $\omega_1,\dots,\omega_n$, and obviously does not vanish.

Hence

$$\theta_0 = \pm 2C;\ \theta_1 = \theta_{-1} = \pm C \neq 0;$$

i.e. the conditions are regular,

If n is odd ($n = 2\mu - 1$), then

$$\theta_0 + \theta_1 s = \pm \begin{vmatrix} 1 & \dots & 1 & (1-s) & 1 & \dots & 1 \\ \omega_1 & \dots & \omega_{\mu-1} & (1-s)\omega_\mu & \omega_{\mu+1} & \dots & \omega_n \\ \cdot & \dots & \cdot & \cdot & \cdot & \dots & \cdot \\ \omega_1^{n-1} & \dots & \omega_{\mu-1}^{n-1} & (1-s)\omega_\mu^{n-1} & \omega_{\mu+1}^{n-1} & \dots & \omega_n^{n-1} \end{vmatrix} = \pm C(1-s).$$

We deduce that in this case, too, the conditions are regular.

(c) *Boundary Conditions for $n = 2$.*

The most general boundary conditions for $n = 2$ have the form

$$\left.\begin{aligned} a_1 y_0' + b_1 y_1' + a_0 y_0 + b_0 y_1 &= 0, \\ c_1 y_0' + d_1 y_1' + c_0 y_0 + d_0 y_1 &= 0. \end{aligned}\right\} \tag{43}$$

We distinguish the following cases:

1. $a_1 d_1 - b_1 c_1 \neq 0$.

By solving (43) for y_0' and y_1', we can bring the conditions to the form

$$\left.\begin{aligned} y_0' + \alpha_{11} y_0 + \alpha_{12} y_1 &= 0, \\ y_1' + \alpha_{21} y_0 + \alpha_{22} y_1 &= 0. \end{aligned}\right\}$$

Then

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \begin{vmatrix} \omega_1 & \omega_2 \\ s\omega_1 & \frac{1}{s}\omega_2 \end{vmatrix} = \frac{1}{s} - s;$$

$$\therefore \theta_0 = 0, \theta_1 = -1, \theta_{-1} = 1,$$

i.e. the conditions are regular.

2. $a_1 d_1 - b_1 c_1 = 0$, $|a_1| + |b_1| > 0$.

In this case we can transform the conditions (43) so:

$$a_1 y_0 + b_1 y_1' + a_0 y_0 + b_0 y_1 = 0,$$

$$c_0 y_0 + d_0 y_1 = 0,$$

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \begin{vmatrix} (a_1 + sb_1)\omega_1 & -\left(a_1 + \frac{1}{s}b_1\right)\omega_1 \\ c_0 + sd_0 & c_0 + \frac{1}{s}d_0 \end{vmatrix}$$

$$= \omega_1(b_1 c_0 + a_1 d_0)\left(s + \frac{1}{s}\right) + 2(a_1, c_0 + b_1 d_0)\omega_1;$$

the conditions are therefore regular if $b_1 c_0 + a_1 d_0 \neq 0$.

3. $a_1 = b_1 = c_1 = d_1 = 0$.

Since the formulae (43) must be independent, in this case $a_0 d_0 - b_0 c_0 \neq 0$; hence the conditions (43) are equivalent to $y_0 = 0, y_1 = 0$. Hence

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \begin{vmatrix} 1 & 1 \\ s & \frac{1}{s} \end{vmatrix} = \frac{1}{s} - s,$$

i.e. the conditions are regular.

Hence *the conditions* (43) *for* $n = 2$ *are regular in just these cases:*

1. $a_1 d_1 - b_1 c_1 \neq 0$;
2. $a_1 d_1 - b_1 c_1 = 0$, $|a_1| + |b_1| > 0$, $b_1 c_0 + a_1 d_0 \neq 0$;
3. $a_1 = b_1 = c_1 = d_1 = 0$, $a_0 d_0 - b_0 c_0 \neq 0$.

9. Asymptotic Behaviour of the Eigenvalues

The results of section 5 enable us to prove the existence of infinitely many eigenvalues for differential operators with regular boundary conditions and to give asymptotic approximation formulae for the behaviour of the sequence of eigenvalues at infinity. It turns out that the principal terms in these approximations are already completely determined by the numbers θ_0,θ_1 or $\theta_0,\theta_1,\theta_{-1}$, which are calculated from the prescribed boundary conditions by means of formulae (38) or (39). The further nature of the differential expression and boundary conditions does not affect them at all.

We assume below (and for brevity we do not mention this again in the hypothesis for the following theorem) that the coefficients of the differential expression considered are continuous in the interval [0,1]. It should, however, be expressly mentioned that all the results of this section would still hold if we required only that these coefficients be summable in the interval [0,1].

THEOREM 2 .

A differential operator of the nth order which is generated by an expression of the reduced form described in section 1 *and by regular boundary conditions has precisely denumerably many*[1] *eigenvalues, whose behaviour at infinity is specified by the following formulae:*

For odd n such that $n = 4\nu - 1$, *the following two sequences are obtained:*[2]

$$\lambda_k' = (-2k\pi i)^n\left\{1 - \frac{n \ln_0\xi^{(1)}}{2k\pi i} + O\left(\frac{1}{k^2}\right)\right\}, \tag{44 a}$$

$$\lambda_k'' = (2k\pi i)^n\left\{1 + \frac{n \ln_0\xi^{(2)}}{2k\pi i} + O\left(\frac{1}{k^2}\right)\right\}, \tag{44 b}$$

$$k = N, N + 1, \ldots;$$

and for odd n such that $n = 4\nu + 1$, *the two sequences:*

$$\lambda_k' = (2k\pi in)^n\left\{1 + \frac{n \ln_0\xi^{(1)}}{2k\pi i} + O\left(\frac{1}{k^2}\right)\right\}, \tag{44 a'}$$

$$\lambda_k'' = (-2k\pi in)^n\left\{1 - \frac{n \ln_0\xi^{(2)}}{2k\pi i} + O\left(\frac{1}{k^2}\right)\right\}, \tag{44 b'}$$

$$k = N, N + 1, \ldots;$$

here $\xi^{(1)}$ *and* $\xi^{(2)}$ *denote the two solutions defined earlier of the equation* $\theta_1\xi + \theta_0 = 0$ *for* S_k *with k odd and k even respectively; and N is a chosen, sufficiently large, positive integer.*

---

[1] In view of the ambiguous usage of the word "denumerable" in English mathematical texts, we have followed the author's terminology.

[2] Here $\ln_0\xi$ is any fixed branch of the natural logarithm.

For even n, $n = 2\mu$, and $\theta_0^2 - 4\theta_1\theta_{-1} \neq 0$, the following sequences are obtained:

$$\lambda_k' = (-1)^\mu(2k\pi)^{2\mu}\left\{1 \mp \frac{\mu \ln_0 \xi'}{k\pi i} + O\left(\frac{1}{k^2}\right)\right\}, \tag{45 a}$$

$$\lambda_k'' = (-1)^\mu(2k\pi)^{2\mu}\left\{1 \mp \frac{\mu \ln_0 \xi''}{k\pi i} + O\left(\frac{1}{k^2}\right)\right\}, \tag{45 b}$$

where ξ' and ξ'' are the roots of the equation

$$\theta_1\xi^2 + \theta_0\xi + \theta_{-1} = 0 \tag{46}$$

for the domain S_0, i.e. are roots of the equation $D_0(\xi) = 0$, and where the upper or lower sign is to be taken according as $n = 4\nu$ or $n = 4\nu + 2$.

For even n, $n = 2\mu$, and $\theta_0^2 - 4\theta_1\theta_{-1} = 0$, the following sequences are obtained:

$$\lambda_k' = (-1)^\mu(2k\pi)^{2\mu}\left\{1 \mp \frac{\mu \ln_0 \xi}{k\pi i} + O\left(\frac{1}{k^{3/2}}\right)\right\}, \tag{47 a}$$

$$\lambda_k'' = (-1)^\mu(2k\pi)^{2\mu}\left\{1 \mp \frac{\mu \ln_0 \xi}{k\pi i} + O\left(\frac{1}{k^{3/2}}\right)\right\}, \tag{47 b}$$

$$k = N, N + 1, \ldots,$$

where ξ is the double root, occurring in this case, of the equation (46) (*relative to the θ for the domain S_0*). *The signs are to be chosen in the same way as for the equations* (45).

In the first three cases, all eigenvalues of sufficiently large modulus are simple; but in the last case, all eigenvalues of sufficiently large modulus can be either simple or double.

Proof. First let n be odd ($n = 2\mu - 1$). We consider a fixed domain T.

Let the numbers ω_k be numbered in the order such that, for $\rho \in T$,

$$\mathscr{R}((\rho + c)\omega_1) \leqslant \mathscr{R}((\rho + c)\omega_2) \leqslant \ldots \leqslant \mathscr{R}((\rho + c)\omega_n). \tag{48}$$

We put

$$\tilde{\rho}_k = (\rho + c)\omega_k, \qquad k = 1, 2, \ldots, n. \tag{49}$$

The points $\tilde{\rho}_1, \tilde{\rho}_2, \ldots, \tilde{\rho}_n$ lie on a circle of radius $|\rho + c|$ and divide the circumference into n equal parts, each corresponding to an angle of $\dfrac{2\pi}{n} = \dfrac{2\pi}{2\mu - 1}$. In the closed, right half-plane there lie certainly at most μ points, since more than μ points (Fig. 10) would lead to the contradiction

$$\pi \geqslant \frac{2\pi}{2\mu - 1}\mu > \pi.$$

Hence, and by the inequalities (48), it follows that at least the first ($\mu - 1$)

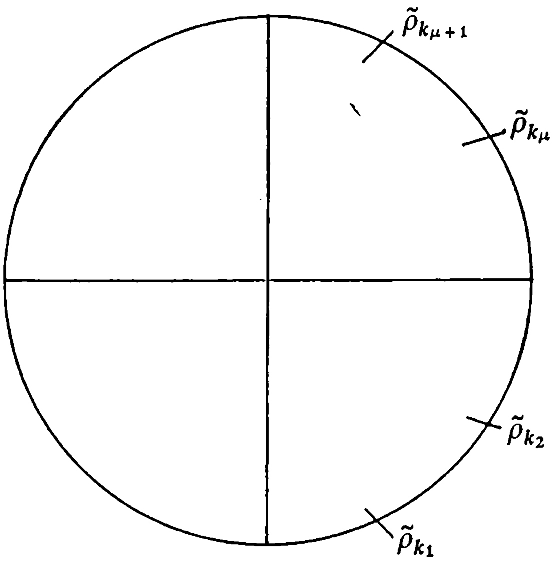

Fig. 10.

points $\tilde{\rho}_1, \tilde{\rho}_2, \ldots, \tilde{\rho}_{\mu-1}$ must lie in the open, left half-plane. Similarly, the last $(\mu - 1)$ points $\tilde{\rho}_{\mu+1}, \tilde{\rho}_{\mu+2}, \ldots, \tilde{\rho}_n$ must lie in the open, right half-plane. In other words, we have:

$$\mathscr{R}(\tilde{\rho}_1) < 0, \mathscr{R}(\tilde{\rho}_2) < 0, \ldots, \mathscr{R}(\tilde{\rho}_{\mu-1}) < 0, \tag{50 a}$$

$$\mathscr{R}(\tilde{\rho}_{\mu+1}) > 0, \mathscr{R}(\tilde{\rho}_{\mu+2}) > 0, \ldots, \mathscr{R}(\tilde{\rho}_n) > 0. \tag{50 b}$$

If $\rho \to \infty$ *and remains in the domain* T, *then the left-hand sides of* (50 a) *and* (50 b) *tend to* $-\infty$ *and* $+\infty$ *respectively.*

For, $\mathscr{R}(\tilde{\rho}_{\mu-1})$ will only then not tend to $-\infty$ if the angular distance between $\tilde{\rho}_{\mu-1}$ and the positive or negative imaginary axis tends to zero. But this would imply that, for sufficiently large $|\rho|$, the $(\mu + 1)$ points $\tilde{\rho}_{\mu-1}, \tilde{\rho}_\mu, \ldots, \tilde{\rho}_n$ lie on the arc

$$-\frac{\pi}{2} - \varepsilon \leqslant \arg \rho \leqslant \frac{\pi}{2} + \varepsilon,$$

implying that, for arbitrarily small $\varepsilon > 0$,

$$\pi + 2\varepsilon \geqslant \frac{2\pi\mu}{2\mu - 1}$$

which is impossible.

From the assertion just proved it follows that, if $\rho \to \infty$, $\rho \in T$, then

(a) $e^{\tilde{\rho}_j}$ *tends exponentially to zero if* $j < \mu$

(b) $e^{\tilde{\rho}_j}$ *tends exponentially to infinity if* $j > \mu$.

By Theorem 1 in section 5 there exists in the domain T a system of n independent solutions $y_1, y_2, \ldots, y_n$ of the equation $l(y) + \rho^n y = 0$ such that

$$\left.\begin{aligned} y_k &= e^{\rho\omega_k x}\left[1 + O\left(\frac{1}{\rho}\right)\right], \\ y_k' &= \rho e^{\rho\omega_k x}\left[\omega_k + O\left(\frac{1}{\rho}\right)\right], \\ y_k^{(n-1)} &= \rho^{n-1} e^{\rho\omega_k x}\left[\omega_k^{n-1} + O\left(\frac{1}{\rho}\right)\right]. \end{aligned}\right\} \tag{51}$$

With these solutions we form the determinant $\Delta(\lambda) = \det[U_\nu(y_j)]$; as was proved in §2.1, the eigenvalues are the zeros of $\Delta(\lambda)$.

So we shall now concern ourselves with this determinant $\Delta(\lambda)$.

For brevity, we introduce the notation $[a] = a + O\left(\frac{1}{\rho}\right)$.

By substituting the expressions (51) into the normalized forms $U_\nu(y)$, we obtain the equations

$$U_{\nu 0}(y_j) = (\rho\omega_j)^{k_\nu}\left\{\alpha_\nu + O\left(\frac{1}{\rho}\right)\right\} = (\rho\omega_j)^{k_\nu}[\alpha_\nu],$$

$$U_{\nu 1}(y_j) = (\rho\omega_j)^{k_\nu} e^{\rho\omega_j}\left\{\beta_\nu + O\left(\frac{1}{\rho}\right)\right\} = (\rho\omega_j)^{k_\nu} e^{\rho\omega_j}[\beta_\nu],$$

and hence

$$U_\nu(y_j) = U_{\nu 0}(y_j) + U_{\nu 1}(y_j) = (\rho\omega_j)^{k_\nu}\{[\alpha_\nu] + e^{\rho\omega_j}[\beta_\nu]\}.$$

If $j < \mu$, the function

$$e^{\rho\omega_j} = e^{-c\omega_j} e^{\tilde{\rho}_j}$$

decreases exponentially as $\rho \to \infty$, $\rho \in T$; hence

$$U_\nu(y_j) = (\rho\omega_j)^{k_\nu}[\alpha_\nu] \text{ for } j < \mu. \tag{52 a}$$

Similarly we find

$$U_\nu(y_j) = (\rho\omega_j)^{k_\nu} e^{\rho\omega_j}[\beta_\nu] \text{ for } j > \mu. \tag{52 b}$$

Finally, we have

$$U_\nu(y_\mu) = (\rho\omega_\mu)^{k_\nu}\{[\alpha_\nu] + e^{\rho\omega_\mu}[\beta_\nu]\}. \tag{52 c}$$

We substitute all these expressions in the equation

$$\Delta = \det[U_\nu(y_j)] = 0 \tag{53}$$

and divide out the common factors $\rho^{k_1}, \rho^{k_2}, \ldots, \rho^{k_n}$ of the rows and also the common factors $e^{\rho\omega_{\mu+1}}, e^{\rho\omega_{\mu+2}}, \ldots, e^{\rho\omega_n}$ of the last μ columns of the determinant $D(\lambda)$. The equation can then be written in the form

$$\Delta_0 = 0,$$

where

$$\Delta_0 = \begin{vmatrix} [\alpha_1]\omega_1^{k_1} & \cdots & [\alpha_1]\omega_{\mu-1}^{k_1} & ([\alpha_1] + [\beta_1]e^{\rho\omega_\mu})\omega_\mu^{k_1} & [\beta_1]\omega_{\mu+1}^{k_1} & \cdots & [\beta_1]\omega_n^{k_1} \\ [\alpha_2]\omega_1^{k_2} & \cdots & [\alpha_2]\omega_{\mu-1}^{k_2} & ([\alpha_2] + [\beta_2]e^{\rho\omega_\mu})\omega_\mu^{k_2} & [\beta_2]\omega_{\mu+1}^{k_2} & \cdots & [\beta_2]\omega_n^{k_2} \\ \cdot & \cdots & \cdot & \cdot & \cdot & & \\ [\alpha_n]\omega_1^{k_n} & \cdots & [\alpha_n]\omega_{\mu-1}^{k_n} & ([\alpha_n] + [\beta_n]e^{\rho\omega_\mu})\omega_\mu^{k_n} & [\beta_n]\omega_{\mu+1}^{k_n} & \cdots & [\beta_n]\omega_n^{k_n} \end{vmatrix}. \tag{54}$$

From the definition of the numbers θ_0 and θ_1 (see (38) in section 8), we therefore have

$$\Delta_0 = [\theta_0] + e^{\rho\omega_\mu}[\theta_1].$$

For a root ρ of the equation $\Delta_0 = 0$ with a sufficiently large absolute value, we have

$$e^{\rho\omega_\mu} = -\frac{[\theta_0]}{[\theta_1]},$$

i.e.
$$e^{\rho\omega_\mu} = -\frac{\theta_0 + O\left(\frac{1}{\rho}\right)}{\theta_1 + O\left(\frac{1}{\rho}\right)} = -\frac{\theta_0}{\theta_1}\left\{1 + O\left(\frac{1}{\rho}\right)\right\} = \xi\left\{1 + O\left(\frac{1}{\rho}\right)\right\}, \tag{55}$$

since, by the regularity of the boundary conditions, $\theta_0 \neq 0$ and $\theta_1 \neq 0$. Hence

$$\rho = \frac{1}{\omega_\mu}\left\{\ln_0\xi + 2k\pi i + O\left(\frac{1}{\rho}\right)\right\}, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{56}$$

We now show that zeros of the function $\Delta(\lambda)$ which are given by formula (56) do actually exist.

We put

$$\rho_k = \frac{1}{\omega_\mu}(2k\pi i + \ln_0\xi); \tag{57}$$

then equation (56) takes the form

$$\rho = \rho_k + O\left(\frac{1}{\rho}\right), \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{58}$$

The numbers ρ_k obviously lie on a parallel to the bisector of the domain T, so that for sufficiently large k there is only one choice possible for the sign of the integer k if the point ρ_k is to lie in T. Closer investigation shows that:

(a_1) $n = 4\nu - 1$.

For a T with even subscript, k must be chosen to be positive; and for a T with odd subscript, k must be negative,

(a$_2$) $n = 4\nu + 1$.

For a T with even subscript, k must be chosen to be negative; and for a T with odd subscript, k must be positive.

About each point ρ_k we now describe a circle Γ_k with the same radius r in each case. By what we have just said, for a sufficiently large $|k|$ these circles will lie entirely within the domain T. The equation (53) is equivalent to equation (55), and since $\xi = e^{\rho_k \omega_\mu}$, the latter can be written in the form

$$e^{\omega_\mu(\rho - \rho_k)} - 1 - O\left(\frac{1}{\rho}\right) = 0. \tag{59}$$

Outside the circles Γ_k *the modulus of the function*

$$f = e^{\omega_\mu(\rho - \rho_k)} - 1 = e^{\omega_\mu(\rho - \rho_0)} - 1$$

is always greater than a positive constant.

For, if we introduce a new variable ζ by putting

$$\omega_\mu(\rho - \rho_0) = \zeta,$$

then

$$f = e^\zeta - 1,$$

and the circles Γ_k transform into circles Γ_k' with the same radius r about the points $\zeta = 2k\pi i$. Since $f = f(\zeta)$ is a periodic function with period $2\pi i$, we have only to show that its modulus is bounded from below by a positive constant in the domain D bounded by the lines $\mathscr{I}\zeta = \pm\pi$ and the circle Γ_0' (Fig. 11). In this domain the function $f(\zeta)$ never vanishes, however, and for sufficiently large $|\mathscr{R}\zeta|$, say $|\mathscr{R}\zeta| > N$, $|f(\zeta)|$ remains greater than a positive constant, because

$$\lim_{\mathscr{R}\zeta \to +\infty} |f(\zeta)| = \infty$$

$$\lim_{\mathscr{R}\zeta \to -\infty} |f(\zeta)| = 1.$$

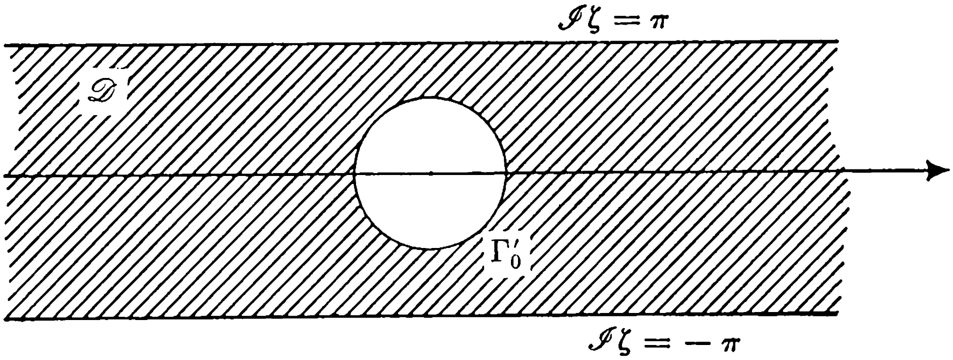

Fig. 11.

This proves our assertion. We conclude from it that, for sufficiently large ρ the function Δ has no zeros outside the circles Γ_k.

Let m be the minimum of the function $|e^{\omega_\mu(\rho-\rho_k)}-1|$ on Γ_k. Since $\rho-\rho_k=re^{i\theta}$ on Γ_k, m does not depend on k. For sufficiently large ρ, $|O(1/\rho)|<m$ on Γ_k.

Hence, by Rouché's theorem (see, e.g. [91]), it follows that, within Γ_k, the equation (59) has the same number of zeros as the equation $e^{\omega_\mu(\rho-\rho_k)}-1=0$, i.e. just one zero, which we denote by ρ_k'.

By (56),

$$\rho_k'=\frac{1}{\omega_\mu}\left\{2k\pi i+\ln_0\xi+O\left(\frac{1}{\rho_k'}\right)\right\};$$

but by the same formula,

$$O\left(\frac{1}{\rho_k'}\right)=O\left(\frac{1}{k}\right).$$

We have therefore

$$\rho_k'=\frac{1}{\omega_\mu}\left\{2k\pi i+\ln_0\xi+O\left(\frac{1}{k}\right)\right\},$$

or

$$\rho_k'=\frac{2k\pi i}{\omega_\mu}\left\{1+\frac{\ln_0\xi}{2k\pi i}+O\left(\frac{1}{k^2}\right)\right\}. \tag{60}$$

If we apply the same arguments to a domain T, first with even subscript, then with odd subscript, and raise the result to the nth power, then, taking account of the proper sign of the number k, we obtain for each case the desired sequence of eigenvalues (44).

That these eigenvalues are simple for sufficiently large k rests on the fact shown above that they are simple zeros of the determinant $\Delta(\lambda)$.

Now let $n=2\mu$, i.e. n is even. We again consider a fixed domain T for which the inequalities (48) hold. By the same considerations as were used for n odd, we conclude that

$$\mathscr{R}(\rho_1)<0,\ \mathscr{R}(\rho_2)<0,\ldots,\mathscr{R}(\rho_{\mu-1})<0, \tag{61}$$

$$\mathscr{R}(\rho_{\mu+2})>0,\ \mathscr{R}(\rho_{\mu+3})>0,\ldots,\mathscr{R}(\rho_n)>0, \tag{62}$$

and the left-hand sides of (61) and (62) tend exponentially to $-\infty$ and $+\infty$ respectively, as $\rho\to\infty$ while remaining in the prescribed domain T.

Hence, we conclude, exactly as in the case of n odd, that in the domain T

$$\left.\begin{aligned}U_\nu(y_j)&=(\rho\omega_j)^{k_\nu}[\alpha_\nu] &&\text{for } j\leqslant\mu-1,\\ U_\nu(y_j)&=(\rho\omega_j)^{k_\nu}e^{\rho\omega_j}[\beta_\nu] &&\text{for } j\geqslant\mu+2,\end{aligned}\right\} \tag{63}$$

and moreover

$$U_\nu(y_\mu)=(\rho\omega_\mu)^{k_\nu}\{[\alpha_\nu]+e^{\rho\omega_\mu}[\beta_\nu]\}, \tag{64 a}$$

$$U_\nu(y_{\mu+1})=(\rho\omega_{\mu+1})^{k_\nu}\{[\alpha_\nu]+e^{\rho\omega_{\mu+1}}[\beta_\nu]\}. \tag{64 b}$$

When $n = 2\mu$, if ω_k is a root of the equation $\omega^n + 1 = 0$, then so is $-\omega_k$. Hence, and by the inequalities (48), it follows that

$$\mathscr{R}(\tilde{\rho}_\mu) \leqslant 0, \qquad \mathscr{R}(\tilde{\rho}_{\mu+1}) \geqslant 0,$$

and

$$\omega_1 = -\omega_n, \, \omega_2 = -\omega_{n-1}, \ldots, \omega_\mu = -\omega_{\mu+1}.$$

Hence (64 b) can be re-written as

$$U_\nu(y_{\mu+1}) = (\rho\omega_{\mu+1})^{k_\nu}\{[\alpha_\nu] + e^{-\rho\omega_\mu}[\beta_\nu]\}. \tag{64 b$'$}$$

By substituting the expressions (63), (64) for $U_\nu(y_j)$ into the equation $\Delta = 0$ and removing the common factors, we obtain an equation of the form

$$\Delta_0 = 0 \tag{65}$$

where $\Delta_0 =$

$$\begin{vmatrix} [\alpha_1\omega_1^{k_1}] \ldots [\alpha_1\omega^{k_1}_{-\mu 1}] & \omega_\mu^{k_1}\{[\alpha_1] + e^{\rho\omega_\mu}[\beta_1]\} & \omega_{\mu+1}^{k_1}\{[\alpha_1] + e^{-\rho\omega_\mu}[\beta_1]\} & [\beta_1\omega_{\mu+2}^{k_1}] \ldots [\beta_1\omega_n^{k_1}] \\ [\alpha_2\omega_1^{k_2}] \ldots [\alpha_2\omega_{\mu-1}^{k_2}] & \omega_\mu^{k_2}\{[\alpha_2] + e^{\rho\omega_\mu}[\beta_2]\} & \omega_{\mu+1}^{k_2}\{[\alpha_2] + e^{-\rho\omega_\mu}[\beta_2]\} & [\beta_2\omega_{\mu+2}^{k_2}] \ldots [\beta_2\omega_n^{k_2}] \\ \cdot \quad \cdots \quad \cdot & \cdot & \cdot & \cdot \quad \cdots \\ [\alpha_n\omega_1^{k_n}] \ldots [\alpha_n\omega_{\mu-1}^{k_n}] & \omega_\mu^{k_n}\{[\alpha_n] + e^{\rho\omega_\mu}[\beta_n]\} & \omega_{\mu+1}^{k_n}\{[\alpha_n] + e^{-\rho\omega_\mu}[\beta_n]\} & [\beta_n\omega_{\mu+2}^{k_n}] \ldots [\beta_n\omega_n^{k_n}] \end{vmatrix} \tag{66}$$

By the definition of the numbers $\theta_0, \theta_1, \theta_{-1}$ (see (39), section 8),

$$\Delta_0 = [\theta_0] + [\theta_1]e^{\rho\omega_\mu} + [\theta_{-1}]e^{-\rho\omega_\mu}$$

and therefore

$$e^{\rho\omega_\mu}\Delta_0 = [\theta_1]e^{2\rho\omega_\mu} + [\theta_0]e^{\rho\omega_\mu} + [\theta_{-1}] = \theta_1 e^{2\rho\omega_\mu} + \theta_0 e^{\rho\omega_\mu} + \theta_{-1} + O\left(\frac{1}{\rho}\right), \tag{67}$$

because, by virtue of the relation $\mathscr{R}(\tilde{\rho}_\mu) \leqslant 0$, we clearly have

$$|e^{\rho\omega_\mu}| = |e^{\tilde{\rho}_\mu}| \, |e^{-c\omega_\mu}| \leqslant |e^{-c\omega_\mu}|,$$

i.e. the function $e^{\rho\omega_\mu}$ is bounded in the domain T.

We consider the quadratic equation

$$\theta_1\xi^2 + \theta_0\xi + \theta_{-1} = 0; \tag{68}$$

let ξ', ξ'' be its roots, so that

$$\theta_1\xi^2 + \theta_0\xi + \theta_{-1} = \theta_1(\xi - \xi')(\xi - \xi'').$$

Then (67) can be rewritten as

$$e^{\rho\omega_\mu}\Delta_0 = \theta_1(e^{\rho\omega_\mu} - \xi')(e^{\rho\omega_\mu} - \xi'') + O\left(\frac{1}{\rho}\right). \tag{69}$$

The equations

$$e^{\rho\omega_\mu} - \xi' = 0 \text{ and } e^{\rho\omega_\mu} - \xi'' = 0$$

have respectively the roots

$$\frac{1}{\omega_\mu}(\ln_0 \xi' + 2k\pi i), \quad \frac{1}{\omega_\mu}(\ln_0 \xi'' + 2k\pi i), \quad k = 0, \pm 1, \pm 2, \ldots,$$

but we are concerned only with those which lie within the domain T.

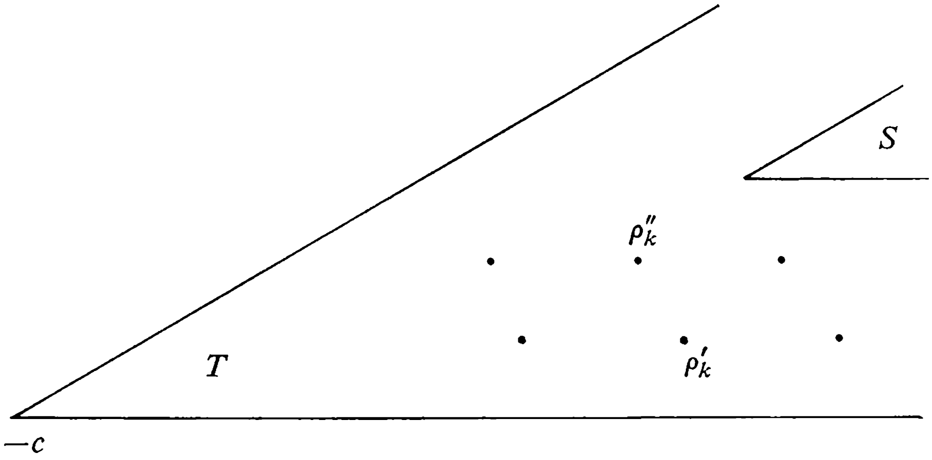

Fig. 12.

Now the roots of both these types obviously lie on a parallel to one or other of the boundary lines of the domain S. Consider the domain S_0, for example. Then we have

$$\omega_\mu = -ie^{-i\pi/n} \text{ for } n = 4\nu,$$

$$\omega_\mu = i \qquad \text{for } n = 4\nu + 2,$$

and the above roots are given by

$$ie^{i\pi/n} \ln_0 \xi' - 2k\pi, \quad ie^{i\pi/n} \ln_0 \xi'' - 2k\pi \quad \text{for } n = 4\nu,$$

and by

$$-i \ln_0 \xi' + 2k\pi, \quad -i \ln_0 \xi'' + 2k\pi \quad \text{for } n = 4\nu + 2.$$

Hence it follows that, for $n = 4\nu$ all roots with $k < 0$, and for $n = 4\nu + 2$ all roots with $k > 0$, are situated inside T_0 and at a positive distance from its boundaries, if T_0 is suitably chosen, i.e. if c is suitably chosen.

In the first of these cases, we replace k by $-k$ and so arrive at the two sequences

$$\rho'_k = \frac{1}{\omega_\mu}(\ln_0 \xi' \mp 2k\pi i), \qquad \rho''_k = \frac{1}{\omega_\mu}(\ln_0 \xi'' \pm 2k\pi i), \tag{70}$$

which, for $k = 1, 2, \ldots$ lie inside T_0, provided that the upper sign is chosen for

$n = 4\nu$ and the lower sign if $n = 4\nu + 2$. Using ρ', ρ'' and (69), the equation $\Delta_0 = 0$ can be put into the form

$$[e^{\omega_\mu(\rho - \rho_k')} - 1]\,[e^{\omega_\mu(\rho - \rho_k'')} - 1] + O\left(\frac{1}{\rho}\right) = 0. \tag{71}$$

About each of the points ρ_k', ρ_k'', $k = 1,2,\ldots$, we describe a circle Γ_k' or Γ_k'' with the same radius r. If we choose r to be sufficiently small, all these circles will lie entirely within T_0. If we apply Rouché's theorem again, as we did for n odd, then it follows that for sufficiently large $|\rho|$, $\rho \in T_0$, the equation $\Delta_0 = 0$ can have zeros only within the circles Γ_k' and Γ_k'', and that it has the same number of zeros, in fact, as the equation

$$[e^{\omega_\mu(\rho - \rho_k')} - 1]\,[e^{\omega_\mu(\rho - \rho_k'')} - 1] = 0 \tag{72}$$

has.

Now suppose that $\theta_0^2 - 4\theta_1\,\theta_{-1} \neq 0$; then $\xi' \neq \xi''$. Consequently the numbers ρ_k' and ρ_k'' are pairwise distinct. Hence, by choice of a sufficiently small common radius r it can be ensured that the circles Γ_k' and Γ_k'' do not touch or intersect each other.

In each of these circles the equation (72), and therefore also the equation $\Delta_0 = 0$ for sufficiently large k, have exactly one zero, which we denote by $\tilde{\rho}_k'$ or $\tilde{\rho}_k''$.

In the circle Γ_k' the modulus of the factor $e^{\omega_\mu(\rho - \rho'')} - 1$ remains greater than a certain positive constant; hence by (71) it follows that in the circle Γ_k' the equation $\Delta = 0$ is equivalent to the equation

$$e^{\omega_\mu(\rho - \rho_k')} - 1 = O\left(\frac{1}{\rho}\right).$$

Hence

$$\tilde{\rho}_k' = \rho_k' + O\left(\frac{1}{\rho}\right),$$

and similarly

$$\tilde{\rho}_k'' = \rho_k'' + O\left(\frac{1}{\rho}\right).$$

As for n odd, this implies that

$$\tilde{\rho}_k' = \mp \frac{2k\pi i}{\omega_\mu}\left\{1 \mp \frac{\ln_0 \xi'}{2k\pi i} + O\left(\frac{1}{k^2}\right)\right\},$$

and similarly for $\tilde{\rho}_k''$.

On raising this to the nth power, we obtain the formulae (45).

It is important to notice here that consideration of a domain T_k with k odd would yield no new eigenvalues. For if we consider, for $n = 4\nu$, instead of the domain T_0 a suitable domain, for example, T_1, then ξ' and ξ'' go over into $1/\xi'$ and $1/\xi''$, and so the logarithms $\ln_0\xi'$ and $\ln_0\xi''$ become $-\ln_0\xi'$ and $-\ln_0\xi''$ respectively, apart from a possible addend $2k'\pi i$. On the other hand, ω_μ is also to be replaced by $-\omega_\mu$, as may easily be seen from the considerations of section 8. Hence, in order that the numbers $\frac{1}{\omega_\mu}(\ln_0 \xi' + 2k\pi i)$ shall lie in T_1, it is now necessary that k be positive, and this implies that in formula (70) for $n = 4\nu$ the plus sign now stands.

Hence the ρ_k' and also the ρ_k'' remain unchanged. Since the equation (71) also remains unchanged (up to multiplication by a constant factor), the same $\tilde{\rho}_k'$ and $\tilde{\rho}_k''$ result. The same thing holds for $n = 4\nu + 2$ if we go over from the domain T_0 to T_{2n-1}. Hence, for large $|\lambda|$, the eigenvalues (45) are the only eigenvalues of the operator L.

Now suppose that $\theta_0^2 - 4\theta_1\theta_{-1} = 0$. Then we have $\xi' = \xi''$, and so $\rho_k' = \rho_k''$, and the circles Γ_k' and Γ_k'' coincide for each k. The equation $\Delta_0 = 0$ therefore has, for sufficiently large $|\rho|$, precisely two zeros in each of these circles $\Gamma_k = \Gamma_k' = \Gamma_k''$, though these may possibly coincide to form a double root.

Let $\tilde{\rho}_k$ be either of these zeros. The equation (71) then takes the form

$$[e^{\omega_\mu(\rho - \rho_k')} - 1]^2 + O\left(\frac{1}{\rho}\right) = 0.$$

It follows that

$$e^{\omega_\mu(\rho - \rho_k')} - 1 + O\left(\frac{1}{\sqrt{|\rho|}}\right) = 0,$$

and hence

$$\tilde{\rho}_k = \rho_k' + O\left(\frac{1}{\sqrt{|\rho|}}\right).$$

The last part of the argument as in the case $\theta_0^2 - 4\theta_1\theta_1 \neq 0$ enables us to derive the formulae (47).

Remark. If the coefficients $p_2(x),\ldots,p_n(x)$ have continuous derivatives up to a definite order, then more precise asymptotic formulae can be derived, involving higher powers of $1/\rho$ (see, for example, [54] and [110]).

10. Asymptotic Behaviour of the Eigenfunctions

We shall now use the results of sections 5 and 9 to elucidate the asymptotic behaviour of the eigenfunctions for eigenvalues with very large moduli.

We again assume that L is a differential operator in the interval $[0,1]$ and is generated by a differential expression with continuous coefficients and by regular boundary conditions.

Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the equation $l(y) + \rho^n y = 0$ which satisfy the relations (17) in a certain domain T (cf. Theorem 1, section 5). An eigenfunction which belongs to a prescribed eigenvalue $\lambda = -\rho^n$ with $\rho \in T$ must be expressible as a linear combination of the functions $y_1, y_2, \ldots, y_n$:

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n,$$

where the coefficients c_j are non-trivial solutions of the system of homogeneous equations

$$U_\nu(y_1)c_1 + U_\nu(y_2)c_2 + \ldots + U_\nu(y_n)c_n = 0, \qquad \nu = 1,2,\ldots,n.$$

For simplicity, we consider only a simple eigenvalue λ, for which the rank of the determinant $\Delta = \det\,[U_\nu(y_k)]$, $\nu,k = 1,\ldots,n$, is equal to $(n-1)$.

Then

$$y = \begin{vmatrix} y_1 & y_2 & & y_n \\ U_2(y_1) & U_2(y_2) & & U_2(y_n) \\ \cdot & \cdot & & \cdot \\ U_n(y_1) & U_n(y_2) & & U_n(y_n) \end{vmatrix} \tag{73}$$

is an eigenfunction belonging to the eigenvalue λ if we assume that not all the minors of the elements of the first row of the determinant Δ vanish. (If the latter were the case, we should have to arrange the $y_1,y_2,\ldots,y_n$ in a row, of which not all the minors vanish).

We examine the cases for n even and n odd separately.

(a) *Let n be odd* $(n = 2\mu - 1)$

In (73) we substitute for y_j and $U_\nu(y_j)$ their representations (17) and (52) and divide through by the inessential factors $\rho^{k_2},\rho^{k_3},\ldots,\rho^{k_n},e^{\rho\omega_{\mu+1}},\ldots,e^{\rho\omega_n}$. Then an eigenfunction y_0 is obtained, perhaps not normalized, for which the following formula holds:

$$y_0 = \begin{vmatrix} e^{\rho\omega_1 x}[1] & \ldots & e^{\rho\omega_{\mu-1}x}[1] & e^{\rho\omega_\mu x}[1] & e^{\rho\omega_{\mu+1}(x-1)}[1] & \ldots & e^{\rho\omega_n(x-1)}[1] \\ [\alpha_2]\omega_1^{k_2} & \ldots & [\alpha_2]\omega_{\mu-1}^{k_2} & ([\alpha_2] + e^{\rho\omega_\mu}[\beta_2]\omega_\mu^{k_2}) & [\beta_2]\omega_{\mu+1}^{k_2} & \ldots & [\beta_2]\omega_n^{k_2} \\ \cdot & \ldots & \cdot & \cdot & \cdot & & \cdot \\ [\alpha_n]\omega_1^{k_n} & \ldots & [\alpha_n]\omega_{\mu-1}^{k_n} & ([\alpha_n] + e^{\rho\omega_\mu}[\beta_n]\omega_\mu^{k_n}) & [\beta_n]\omega_{\mu+1}^{k_n} & \ldots & [\beta_n]\omega_n^{k_n} \end{vmatrix} \tag{74}$$

Now for sufficiently large $|\lambda|$ the number ρ must correspond to one of the ρ_k' or ρ_k'' lying in T; we therefore obtain for a domain T with an odd subscript

$$\rho = \rho_k' = \frac{1}{\omega_\mu}\left[\mp 2k\pi i + \ln_0 \xi^{(1)} + O\left(\frac{1}{k}\right)\right],$$

and for a domain T with even subscript

$$\rho = \rho_k'' = \frac{1}{\omega_\mu}\left[\pm 2k\pi i + \ln_0 \xi^{(2)} + O\left(\frac{1}{k}\right)\right]$$

for a suitable positive k; and in these formulae the upper or lower sign is to be taken according as $n = 4\nu - 1$ or $n = 4\nu + 1$. We are therefore led to define

$$\rho_k^{(1)} = \frac{1}{\omega_\mu}(\mp 2k\pi i + \ln_0 \xi^{(1)}), \qquad \rho_k^{(2)} = \frac{1}{\omega_\mu}(\pm 2k\pi i + \ln_0 \xi^{(2)}),$$

in each case; and we then have

$$\rho_k' = \rho_k^{(1)} + O\left(\frac{1}{k}\right), \qquad \rho_k'' = \rho_k^{(2)} + O\left(\frac{1}{k}\right).$$

Then obviously

$$e^{\rho\omega_\mu x} = e^{\rho_k'\omega_\mu x} = e^{\omega_\mu \rho_k^{(1)} x}[1],$$

since

$$e^{O(\frac{1}{k})} = 1 + O\left(\frac{1}{k}\right) + \left(O\left(\frac{1}{k}\right)\right)^2 + \ldots = 1 + O\left(\frac{1}{k}\right) = [1].$$

In particular,

$$e^{\rho\omega_\mu} = e^{\pm 2k\pi i + \ln_0 \xi}[1] = \xi[1].$$

By substituting these expressions in (74) we obtain the following representation for each of the eigenfunctions $y_k^{(1)}$, $y_k^{(2)}$ belonging to the eigenvalues λ_k', λ_k'' of formula (44)

$$y_k^{(\sigma)} =$$

$$\begin{vmatrix} e^{\omega_1 \rho_k^{(\sigma)} x}[1] & \ldots & e^{\omega_{\mu-1} \rho_k^{(\sigma)} x}[1] & e^{\omega_\mu \rho_k^{(\sigma)} x}[1] & e^{\omega_{\mu+1} \rho_k^{(\sigma)} (x-1)}[1] & \ldots & e^{\omega_n \rho_k^{(\sigma)} (x-1)}[1] \\ [\alpha_2]\omega_1^{k_2} & \ldots & [\alpha_2]\omega_{\mu-1}^{k_2} & [\alpha_2 + \xi\beta_2]\omega_\mu^{k_2} & [\beta_2]\omega_{\mu+1}^{k_2} & \ldots & [\beta_2]\omega_n^{k_2} \\ \cdot & \ldots & \cdot & \cdot & \cdot & & \\ [\alpha_n]\omega_1^{k_n} & \ldots & [\alpha_n]\omega_{\mu-1}^{k_n} & [\alpha_n + \xi\beta_n]\omega_\mu^{k_n} & [\beta_n]\omega_{\mu+1}^{k_n} & \ldots & [\beta_n]\omega_n^{k_n} \end{vmatrix} \tag{75}$$

$$k = N, (N+1), \ldots, \qquad \sigma = 1,2.$$

If we restrict ourselves to a closed sub-interval $x' \leqslant x \leqslant x''$ inside the interval [0,1], formula (75) can be simplified considerably. For, if we take $e^{\omega_\mu \rho_k^{(\sigma)} x}$ outside the determinant, then the μth element of the first row is [1], and the other elements of this row become:

$$e^{(\omega_\nu - \omega_\mu)\rho_k^{(\sigma)} x}[1] \text{ for } \nu < \mu \text{ and } e^{(\omega_\nu - \omega_\mu)\rho_k^{(\sigma)}(x-1)}[1] \text{ for } \nu > \mu,$$

and these expressions tend exponentially to zero when $k \to \infty$.
Hence

$$y_k^{(\sigma)} = (-1)^{\mu-1} e^{\omega_\mu \rho_k^{(\sigma)} x} \left\{ \begin{vmatrix} \alpha_2\omega_1^{k_2} & \ldots & \alpha_2\omega_{\mu-1}^{k_2} & \beta_2\omega_{\mu+1}^{k_2} & \ldots & \beta_2\omega_n^{k_2} \\ & & \cdot & \cdot & & \\ \alpha_n\omega_1^{k_n} & \ldots & \alpha_n\omega_{\mu-1}^{k_n} & \beta_n\omega_{\mu+1}^{k_n} & \ldots & \beta_n\omega_n^{k_n} \end{vmatrix} + O\left(\frac{1}{\rho}\right) \right\}. \tag{76}$$

It must be noted here that, in general, the numerical order of the numbers $\omega_1, \ldots, \omega_n$ will be different for the sequences $y_k^{(1)}$ and $y_k^{(2)}$.

(b) *Let n be even* ($n = 2\mu$).

We repeat the above argument and obtain two sequences of eigenfunctions corresponding to the eigenvalues λ_k' and λ_k''. To the eigenvalue λ_k' corresponds the eigenfunction

$$y_{k_1} = \begin{vmatrix} e^{\omega_1\rho_k' x}[1] & \ldots & e^{\omega_{\mu-1}\rho_k' x}[1] & e^{\omega_\mu\rho_k' x}[1] & e^{-\omega_\mu\rho_k' x}[1] & e^{\omega_{\mu+2}\rho_k'(x-1)}[1] & \ldots e^{\omega_n\rho_k'(x-1)}[1] \\ [\alpha_2]\omega_1^{k_2} & \ldots & [\alpha_2]\omega_{\mu-1}^{k_2} & [\alpha_2 + \xi'\beta_2]\omega_\mu^{k_2} & \left[\alpha_2 + \frac{1}{\xi'}\beta_2\right]\omega_{\mu+1}^{k_2} & [\beta_2]\omega_{\mu+2}^{k_2} & \ldots[\beta_2]\omega_n^{k_2} \\ \cdot & \cdot & & \cdot & \cdot & & \\ [\alpha_n]\omega_1^{k_n} & \ldots & [\alpha_n]\omega_{\mu-1}^{k_n} & [\alpha_n + \xi'\beta_n]\omega_\mu^{k_n} & \left[\alpha_n + \frac{1}{\xi'}\beta_n\right]\omega_{\mu+1}^{k_n} & [\beta_n]\omega_{\mu+2}^{k_n} & \ldots[\beta_n]\omega_n^{k_n} \end{vmatrix} \tag{77}$$

$$k = 1,2,3,\ldots .$$

For any closed sub-interval of the interval [0,1] this formula simplifies to

$$y_{k_1} = (-1)^{\mu-1}e^{\omega_\mu\rho_k' x}\left\{\begin{vmatrix} \alpha_2\omega_1^{k_2} & \ldots & \alpha_2\omega_{\mu-1}^{k_2} & \left(\alpha_2 + \frac{1}{\xi'}\beta_2\right)\omega_{\mu+1}^{k_2} & \beta_2\omega_{\mu+2}^{k_2} & \ldots & \beta_2\omega_n^{k_2} \\ & & & \cdot & & & \\ \alpha_n\omega_1^{k_n} & \ldots & \alpha_n\omega_{\mu-1}^{k_n} & \left(\alpha_n + \frac{1}{\xi'}\beta_n\right)\omega_{\mu+1}^{k_n} & \beta_n\omega_{\mu+2}^{k_n} & \ldots & \beta_n\omega_n^{k_n} \end{vmatrix} + O\left(\frac{1}{k}\right)\right\}$$

$$+(-1)^{\mu}e^{-\omega_\mu\rho_k' x}\left\{\begin{vmatrix} \alpha_2\omega_1^{k_2} & \ldots & \alpha_2\omega_{\mu-1}^{k_2} & (\alpha_2 + \xi'\beta_2)\omega_\mu^{k_2} & \beta_2\omega_{\mu+2}^{k_2} & \ldots & \beta_2\omega_n^{k_2} \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \alpha_n\omega_1^{k_n} & \ldots & \alpha_n\omega_{\mu-1}^{k_n} & (\alpha_n + \xi'\beta_n)\omega_\mu^{k_n} & \beta_n\omega_{\mu+2}^{k_n} & \ldots & \beta_n\omega_n^{k_n} \end{vmatrix} + O\left(\frac{1}{k}\right)\right\} \tag{78}$$

The formula for the eigenfunction y_{k_2} corresponding to the eigenvalue λ_k'' is obtained by replacing ρ_k' and ξ' by ρ_k'' and ξ''.

For $n = 2$, the formulae are particularly simple. If, for example, the domain T is the first quadrant of the ρ-plane, then we have $\omega_\mu = i$ and $\omega_{\mu+1} = -i$, and formula (77) then has the form

$$y_{k_1} = (-i)^{k_2}e^{i\rho_k' x}\left[\alpha_2 + \frac{1}{\xi'}\beta_2 + O\left(\frac{1}{k}\right)\right] - (i)^{k_2}e^{-i\rho_k' x}\left[\alpha_2 + \xi'\beta_2 + O\left(\frac{1}{k}\right)\right].$$

The results presented in this section are derived essentially from Birkhoff's work [5]; the method of proving Theorem 1 stems from Stone [109a].

Theorem 1 has been proved by Birkhoff under more general conditions; viz. the coefficients p_k, $k = 2,3,\ldots,n$ can be analytic functions of the parameter ρ which are representable in the neighbourhood of infinity by

$$p_k(x,\rho) = \sum_{\nu=0}^{\infty} a_{k\nu}(x)\rho^{-\nu}$$

(and this representation may even be taken to hold in the sense of asymptotic convergence). The domains S and T also become more general, and instead of formula (17) the following formula is obtained:

$$y_k = \left[u_k(x) + O\left(\frac{1}{\rho}\right)\right] \exp\left\{\rho \int_0^x \omega_k(t)dt\right\},$$

where $\omega_1(x),\ldots,\omega_n(x)$ are the roots of the equation

$$\omega^n + a_{n-1,0}(x)\omega^{n-1} + \ldots + a_{0,0}(x) = 0,$$

and $u_k(x)$ are certain functions of x which are defined by formal substitution for y_k in the differential equation.

11. Various Generalizations of the Asymptotic Estimates

(a) *Arbitrary Interval.*

An operator L defined in an arbitrary interval $[a,b],a < b$, can always be brought, by the transformation $x = a + t(b - a)$, into a form for which the above asymptotic formulae are applicable, provided only that the differential expression for L satisfies the conditions required in section 1. Correspondingly, in the formulae (17) and (75) to (78) x is to be replaced by $(x - a)/(b - a)$, and in the formulae (44) to (47) for the eigenvalues the number π must be replaced by $\pi/(b - a)$.

(b). *The Boundary-value Problem $L(y) = \lambda\rho y$.*

Let L be an arbitrary differential operator for the interval $[a,b]$ and in whose differential expression $l(y)$ the coefficient $p_0(x)$ is $\equiv 1$. Suppose that the function $\rho(x)$ is real and continuous and has the same sign throughout $[a,b]$; without loss of generality, we may take $p(x) > 0$, for if this is not so initially, we have only to replace ρ and λ by $-\rho$ and $-\lambda$. We introduce a new independent variable t by the substitution

$$t = \frac{1}{h}\int_a^x \sqrt[n]{\rho(\xi)}\, d\xi, \tag{79}$$

where

$$h = \int_a^b \sqrt[n]{\rho(\xi)}\, d\xi. \tag{80}$$

With the variable t, the boundary-value problem $L(y) = \lambda\rho y$ can be written in the form $L_1 y = \lambda y$, where L_1 is the operator which is obtained from L by the change of variable to t and division by $\rho(x)$. For its differential expression the coefficient of the highest derivative is again equal to 1.

Then by a change of the dependent variable to $\tilde{y}$, where

$$y = \tilde{y} \exp\left(-\frac{1}{n}\int p_1(x)dx\right),$$

we obtain finally from L_1 an operator L_2 for which all the assumptions of Theorem 2 are satisfied and accordingly the previous asymptotic estimates (17), (44)–(47), (75)–(78) may legitimately be applied, provided that the condition for regularity of the boundary-conditions, and the incidental assumption that the eigenvalues are simple, are satisfied. If we now transform these formulae back to the old variables, we obtain the corresponding formulae for the boundary-value problem $L(y) = \lambda\rho y$. In particular, the formulae for the eigenvalues are obtained from (44) to (47) by replacing π by $\pi/\{\int_a^b \sqrt[n]{\rho(x)}\, dx\}$.

Obviously we have also arrived at asymptotic approximations for any eigenvalue problem $L(y) = \lambda y$ of the type considered in Chapter 1, for which the condition $p_0(x) = 1$ is unnecessary.

The important assumption, of course, here is that $p_0(x)$ is real in $[a,b]$, does not vanish, and does not change sign.

Considerable complications arise in the case where the function $\rho(x)$ vanishes at some points of the interval $[a,b]$, or, what comes to the same thing, if the coefficient $p_0(x)$ in an eigenvalue problem $L(y) = \lambda y$ vanishes at certain points of (a,b). This case and some of its various generalizations were the subject of investigations by R. Langer and other authors (see the paper by R. Langer [57e], which deals with the case $n = 2$).

The most general results in this direction have been obtained by V. S. Pugachev [92].

(c). J. D. Tamarkin [110] obtains asymptotic expressions for the eigenvalues of the generalized boundary-value problem

$$y^{(n)} + p_1(x,\lambda)y^{(n-1)} + \dots + p_n(x,\lambda)y = 0; \quad U_\nu(y) = 0, \ \nu = 1,2,\dots,n,$$

where $p_k(x,\lambda)$ is a polynomial of the kth degree in λ, and the coefficients of the forms $U_\nu(y)$ are polynomials of the nth degree in λ.

Even more general results (for differential operators with partial derivatives) have been obtained by M. V. Keldysh [47].

§5. *Expansion in Terms of Eigenfunctions*

1. The Basis of the Fourier Method

The solution of partial differential equations by Fourier's method leads to a most important problem: the expansion of a given function in terms of the eigenfunctions of differential operators.

Suppose, for example, that a solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^n u}{\partial x^n} + p_1(x)\frac{\partial^{n-1} u}{\partial x^{n-1}} + \dots + p_n(x)u, \qquad a \leqslant x \leqslant b, \tag{1}$$

is to be found which shall satisfy the initial conditions

$$[u]_{t=0} = f(x); \qquad \left[\frac{\partial u}{\partial t}\right]_{t=0} = \varphi(x); \tag{2}$$

and the boundary conditions

$$\sum_{\nu=0}^{n-1} \alpha_{j\nu}\left(\frac{\partial^\nu u}{\partial x^\nu}\right)_{x=a} + \sum_{\nu=0}^{n-1} \beta_{j\nu}\left(\frac{\partial^\nu u}{\partial x^\nu}\right)_{x=b} = 0, \quad j = 1,2,\dots,n. \tag{3}$$

We shall seek a solution of equation (1) which satisfies the boundary conditions (3) and has the form

$$u = y(x)\,(A\cos pt + B\sin pt). \tag{4}$$

By substitution in (1) and (3), we find that the function $y(x)$ must satisfy the differential equation

$$l(y) \equiv \frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + p_n(x)y = -p^2 y \tag{5}$$

and the boundary conditions

$$U_j(y) \equiv \sum_{\nu=0}^{n-1} \alpha_{j\nu} y_a^{(\nu)} + \sum_{\nu=0}^{n-1} \beta_{j\nu} y_b^{(\nu)} = 0. \tag{6}$$

If now $y \not\equiv 0$, then y is an eigenfunction of the boundary-value problem (5), (6) for the eigenvalue $-p^2$. Let

$$-p_1^2, -p_2^2, -p_3^2, \dots$$

be all the eigenvalues of this problem and

$$y_1(x),\ y_2(x),\ y_3(x), \dots$$

the corresponding eigenfunctions, where each eigenvalue is repeated as many times as it has linearly independent eigenfunctions belonging to it. Then the infinite series

$$u = \sum_{u=1}^{\infty} y_n(x)\,(A_n\cos p_n t + B_n\sin p_n t)$$

satisfies, at least formally, the equation (1) and the boundary conditions (3).

It still has to satisfy the initial conditions. Substitution into the first of the initial conditions yields

$$f(x) = \sum_{n=1}^{\infty} A_n y_n(x). \tag{7}$$

This equation represents a series expansion of the prescribed function $f(x)$ in terms of eigenfunctions of the boundary-value problem.

The question as to the precise basis of the Fourier method thus leads directly to the following problem: *under what conditions may a prescribed function $f(x)$ be expanded as a series of eigenfunctions of a proposed boundary-value problem?*

The solution of this problem is easiest in the case of a self-adjoint boundary-value problem, i.e. if the expression $l(y)$ and the boundary conditions $U_j(y) = 0$, $j = 1,2,\ldots,n$, generate a self-adjoint differential operator.

2. The Case of a Self-Adjoint Operator

Let L be a self-adjoint operator which is developed from a differential expression $l(y)$ and the boundary conditions $U_j(y) = 0, j = 1,2,\ldots,n$. Without loss of generality we can assume that $Ly = 0$ only for $y = 0$, i.e. that the boundary-value problem

$$l(y) = 0, \qquad U_j(y) = 0, \qquad j = 1,2,\ldots,n, \tag{8}$$

has only the trivial solution $y \equiv 0$. For, otherwise, we have only to replace $l(y)$ by the expression $l(y) - cy$, where c is a real number different from all the eigenvalues of the operator L. And we know that such a number exists because a self-adjoint operator can have only denumerably many eigenvalues.

If the boundary-value problem (8) has only the trivial solution, then the operator L has a Green's function (§3, sections 1–6), and this is a Hermitian kernel.

We consider an arbitrary function $f(x)$ in the domain of definition of the operator L: this means that the function $f(x)$ has continuous derivatives up to the nth order and that it satisfies the boundary conditions (6). If we put

$$Lf = h,$$

then

$$f(x) = \int_a^b G(x,\xi)h(\xi)d\xi,$$

i.e. the function $f(x)$ is "sourcewise" representable[1] by the continuous kernel $G(x,\xi)$.

By reason of the Hilbert-Schmidt expansion theorem in the theory of integral equations (see [86a], for example), the function $f(x)$ can be expanded in a uniformly convergent series of eigenfunctions of the kernel $G(x,\xi)$. But we have seen that the kernel $G(x,\xi)$ and the operator L have one and the same set of eigenfunctions. This proves the following theorem.

---

[1] This terminology is commonly used by Russian authors where a function is expressed using a Green's function as kernel.

THEOREM 1

Any function in the domain of definition of a self-adjoint differential operator can be expanded in a uniformly convergent, generalized Fourier series in terms of eigenfunctions of this operator.

We remind ourselves that the domain of definition of a differential operator of the nth order consists of all functions which have continuous derivatives up to the nth order inclusive and which satisfy the boundary conditions belonging to the operator. The above assertion about expansion in eigenfunctions therefore applies to all functions having these properties.

Since the domain of definition of the operator L is dense in $L^2(a,b)$, we deduce from Theorem 1 the immediate

COROLLARY

The eigenfunctions of a self-adjoint differential operator form a complete system in $L^2(a,b)$.

If the eigenfunctions $y_1(x), y_2(x), \ldots$ are chosen so that they form an orthonormal system, then the coefficients A_n in (7) are determined by the formulae

$$A_n = (f,y_n) = \int_a^b f(x)\overline{y_n(x)}dx, \qquad n = 1,2,3,\ldots.$$

These coefficients A_n satisfy *Parseval's relation*

$$\int_a^b |f(x)|^2 dx = \sum_{n=1}^{\infty} |A_n|^2.$$

Theorem 1 remains true in the case of the general boundary-value problem

$$l(y) - \lambda\rho(x)y = 0, \qquad U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n,$$

where $l(y)$ is self-adjoint and the boundary conditions are self-adjoint, provided that the function $\rho(x)$ is continuous and positive in the interval $[a,b]$. For, in this case, the eigenvalue problem $L(y) = \lambda\rho y$ is equivalent to

$$y(x) = \lambda \int_a^b G(x,\xi)\rho(\xi)y(\xi)d\xi,$$

and the integral operator occurring here has the kernel $G_1(x,\xi) = G(x,\xi)\rho(\xi)$ and is a Hermitian operator in the space $\mathscr{R}$ of the functions $f(x)$ with the scalar product

$$(f_1,f_2) = \int_a^b f_1(x)\overline{f_2(x)}\rho(x)dx, \tag{9}$$

i.e. $G_1(x,\xi)$ is a so-called *symmetrizable kernel.*

So the Fourier coefficients A_n are now given by the formulae

$$A_n = (f,y_n) = \int_a^b f(x)\overline{y_n(x)}\rho(x)dx,$$

where the $y_1(x), y_2(x), \ldots$ form a complete system of eigenfunctions of the operator $\frac{1}{\rho}L$, orthonormal in the sense of the scalar product (9).

Kamke [45] weakened the condition $\rho(x) > 0$, replacing it by $\rho(x) \geqslant 0$ with another restriction. He also generalized Theorem 1 to the case of a self-adjoint boundary-value problem

$$l_1(y) + \lambda l_2(y) = 0 \text{ for any real } \lambda,\ U_\nu(y) = 0,\ \nu = 1,2,\ldots,n,$$

under certain additional assumptions relative to the differential expression $l_2(y)$ and the boundary conditions.

3. Expansion in terms of Functions of a Differential Operator with Regular Boundary Conditions

The considerations in §5.2 do not attain the objective if the operator L is not self-adjoint. So we shall now adopt another method which depends on the analytic properties of the operator $L - \lambda 1$ and on the asymptotic formulae obtained in §4. We shall assume that the differential expression from which L is generated has the special form described in §4.1 and that the boundary conditions are regular in the sense of the definition laid down in §4; we shall further assume[1], as in §5.2, that $\lambda = 0$ is not an eigenvalue of the operator, and therefore L has a Green's function $G(x,\xi)$.

In the complex λ-plane we consider a sequence of circles Γ_k $(k = 1,2,\ldots)$ with the origin as their common centre, and having the following properties:

1°. The radius R_k of the circle Γ_k increases without limit as $k \to \infty$.

2°. There is a positive number δ such that all the eigenvalues of the operator lie at a distance $\geqslant \delta$ from each of the circles Γ_k.

By virtue of the asymptotic properties of the eigenvalues proved in §4.9 such circles Γ_k do exist. Let $G(x,s,\lambda)$ be the Green's function of the operator $L - \lambda 1$, and in particular, let $G(x,s,0) = G(x,s)$ be the Green's function for the operator L.

We apply the theorem on residues to the integral

$$I_k = \frac{1}{2\pi i}\oint_{\Gamma_k} \frac{G(x,s,\lambda)d\lambda}{\lambda}$$

and obtain

$$I_k = G(x,s) + \sum_{\nu=1}^{m_k} \frac{H_\nu(x,s)}{\lambda_\nu}, \tag{10}$$

---

[1] It follows from the argument at the beginning of §5.2, and from the asymptotic formulae for the eigenvalues that this assumption does not restrict the generality.

where $H_\nu(x,s)$ is the residue of the function $G(x,s,\lambda)$ at the pole λ_ν and m_k is the number of such poles in the circle Γ_k.

We shall prove that

$$\lim_{k\to\infty} I_k = 0 \tag{11}$$

uniformly with respect to x and s in the interval $[a,b]$. The equation (10) will then lead to the series expansion

$$G(x,s) = -\sum_{\nu=1}^{\infty} \frac{H_\nu(x,s)}{\lambda_\nu}$$

with uniform convergence for all x and s in $[a,b]$.

In fact, it follows from the asymptotic formulae for the eigenfunctions that the circles Γ_k can be chosen so that $m_{k+1} = m_{k+2}$; on the other hand, it is easily seen that, in the square $a \leqslant x,s \leqslant b$,

$$\frac{H_k(x,s)}{\lambda_k} \text{ tends uniformly to zero as } k \to \infty.$$

The proof of the relation (11) is based on

LEMMA 1

On the circumference Γ_k the function $G(x,s,\lambda)$ satisfies the inequality

$$|G(x,s,\lambda)| \leqslant \frac{M}{|\lambda|^{1-1/n}}, \tag{12}$$

where M is a certain constant.

Proof. We put $\lambda = -\rho^n$. Then, with a suitable choice of $\arg\rho$ the circumference Γ_k transforms into the arc γ_k: $0 \leqslant \arg\rho \leqslant 2\pi/n$, $|\rho| = \sqrt[n]{|\lambda|}$; γ_k varies entirely within the two neighbouring domains S_0 and S_1 of the complex ρ-plane.

In order to prove the inequality (12), we make use of formulae (32)–(34) of §3.7; it is convenient to deal with the cases n even and n odd separately.

(a) *Let n be odd*; $n = 2\mu - 1$.

Let the numbers $\omega_1,\omega_2,\ldots,\omega_n$ be numbered in the order such that, for $\rho \in S_0$,

$$\mathscr{R}(\rho\omega_1) \leqslant \mathscr{R}(\rho\omega_2) \leqslant \ldots \leqslant \mathscr{R}(\rho\omega_n).$$

Then, for $\rho \in S_0$, we have

$$\left.\begin{aligned} &\mathscr{R}(\rho\omega_1) \leqslant 0,\ldots,\mathscr{R}(\rho\omega_{\mu-1}) \leqslant 0,\\ &\mathscr{R}(\rho\omega_{\mu+1}) \geqslant 0,\ldots,\mathscr{R}(\rho\omega_n) \geqslant 0.\end{aligned}\right\} \tag{13}$$

We denote the two parts of the arc γ_k which lie in S_0 and for which

$$\mathscr{R}(\rho\omega_\mu) \leqslant 0 \quad \text{or} \quad \mathscr{R}(\rho\omega_\mu) \geqslant 0$$

by γ_k' and γ_k'' respectively (see Fig. 13).

We evaluate the function $G(x,\xi,\lambda)$ on the arc γ_k', making use of the formulae (32)–(36) of §3.7.

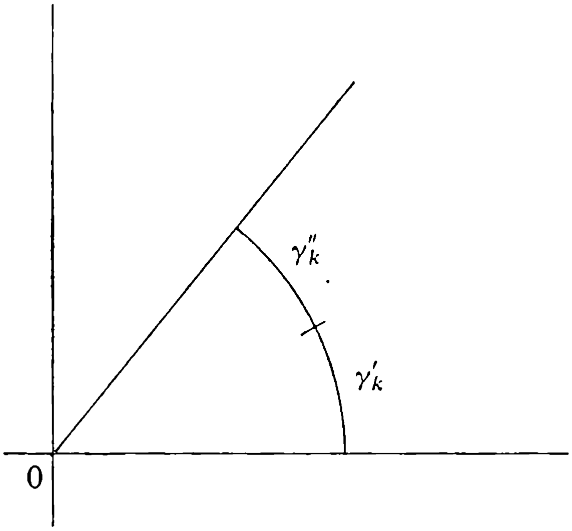

Fig. 13.

We denote by W_ν the cofactor of the element $y_\nu^{(n-1)}$. in the determinant

$$W = \begin{vmatrix} y_1^{(n-1)}(\xi) & y_2^{(n-1)}(\xi) & \cdots & y_n^{(n-1)}(\xi) \\ y_1^{(n-2)}(\xi) & y_2^{(n-2)}(\xi) & \cdots & y_n^{(n-2)}(\xi) \\ \cdot & \cdot & \cdots & \cdot \\ y_1(\xi) & y_2(\xi) & & y_n(\xi) \end{vmatrix}$$

and put

$$z_\nu(\xi) = \frac{W_\nu(\xi)}{W(\xi)}. \tag{14}$$

Then formula (32) of §3.7 can be written as

$$g(x,\xi) = \pm\tfrac{1}{2} \sum_{\nu=1}^{n} y_\nu(x) z_\nu(\xi). \tag{15}$$

By Theorem 1 of §4.5, we have, for $\rho \in S_0$,

$$y_j^{(\nu)}(\xi) = e^{\rho\omega_j\xi}\rho^\nu[\omega_j^\nu],\ j = 1,2,\ldots,n;\ \nu = 0,1,2,\ldots,(n-1).$$

We substitute these expressions into (14) and cancel the factors

$\rho,\rho^2,\dots,\rho^{n-2}$, $e^{\rho\omega_1\xi}$, $e^{\rho\omega_2\xi},\dots,e^{\rho\omega_n\xi}$ in the numerator and denominator, so obtaining

$$z_\nu(\xi) = e^{-\rho\omega_\nu\xi}\frac{1}{\rho^{n-1}}\frac{\beta_\nu}{\beta}, \tag{16}$$

where

$$\beta = \begin{vmatrix} \omega_1^{n-1} & \omega_2^{n-1} & & \omega_n^{n-1} \\ \omega_1^{n-2} & \omega_2^{n-2} & & \omega_n^{n-2} \\ \cdot & \cdot & & \cdot \\ 1 & 1 & \dots & 1 \end{vmatrix},$$

and β_ν is the cofactor of the element ω_ν^{n-1} in this determinant. Hence

$$\sum_{\nu=1}^{n} \omega_\nu^j \frac{\beta_\nu}{\beta} = \begin{cases} 0 \text{ for } j = 0,1,2,\dots,(n-2), \\ 1 \text{ for } j = (n-1). \end{cases} \tag{17}$$

The system (17) has a unique solution, for otherwise it would be satisfied by

$$\frac{\beta_\nu}{\beta} = -\frac{\omega_\nu}{n},$$

since $\omega_\nu^n = -1$. Consequently (16) takes the form

$$z_\nu(\xi) = e^{-\rho\omega_\nu\xi}\frac{1}{n\rho^{n-1}}[-\omega_\nu], \qquad \nu = 1,2,\dots,(n-1). \tag{18}$$

From formula (15) and formula (32) of §3 it follows that

$$U_\nu(g) = -\tfrac{1}{2}\sum_{j=1}^{n} U_{\nu0}(y_j)z_j(\xi) + \tfrac{1}{2}\sum_{j=1}^{n} U_{\nu1}(y_j)z_j(\xi). \tag{19}$$

We consider the function $G(x,\xi,\lambda)$ for $x > \xi$; then the sign in (15) must be $+$. We multiply the first μ columns of the determinant $H(x,\xi,\lambda)$ in the numerator of formula (34) of §3.7 respectively by $\frac{1}{2}z_1(\xi)$, $\frac{1}{2}z_2(\xi),\dots,\frac{1}{2}z_\mu(\xi)$, and also the $(\mu+1)$th to the nth columns by $-\frac{1}{2}z_{\mu+1}(\xi), -\frac{1}{2}z_{\mu+2}(\xi),\dots,-\frac{1}{2}z_n(\xi)$ respectively, and then add all to the last column. The elements of the last column then have the form

$$\sum_{j=1}^{\mu} y_j(x)z_j(\xi) \qquad \text{in the first row,}$$

and

$$\sum_{j=1}^{\mu} U_{\nu1}(y_j)z_j(\xi) - \sum_{j=\mu+1}^{n} U_{\nu0}(y_j)z_j(\xi), \qquad \nu = 1,2,\dots,n,$$

in the 2nd to $(n+1)$th rows. By formulae (17) and (52) of §4 and formula (18) above, these elements can be written in the following form, with the introduction of $P_\nu, \nu = 0,1,\ldots,n$,

$$\frac{1}{n\rho^{n-1}} P_0 = -\frac{1}{n\rho^{n-1}} \sum_{\nu=1}^{\mu} e^{\rho\omega_\nu(x-\xi)}[\omega_\nu],$$

$$\frac{\rho^{k_\nu}}{n\rho^{n-1}} P_\nu = \frac{1}{n\rho^{n-1}} \left\{ \sum_{j=1}^{\mu} e^{\rho\omega_j(1-\xi)} \rho^{k_\nu}[-\beta_\nu \omega_j^{k_\nu+1}] + \sum_{j=\mu+1}^{n} e^{-\rho\omega_j\xi} \rho^{k_\nu}[\alpha_\nu \omega_j^{k_\nu+1}] \right\}.$$

Moreover, we have the asymptotic formulae proved in §4.5 and 4.9:

$$y_\nu = e^{\rho\omega_\nu x}[1], \qquad \nu = 1,2,\ldots,n,$$

$$U_\nu(y_j) = \begin{cases} \rho^{k_\nu}[\alpha_\nu \omega_j^{k_\nu}] & \text{for } j = 1,2,\ldots,(\mu-1), \\ \rho^{k_\nu}([\alpha_\nu \omega_\mu^{k_\nu}] + e^{\rho\omega_\mu}[\beta_\nu \omega_\mu^{k_\nu}]) & \text{for } j = \mu, \\ \rho^{k_\nu} e^{\rho\omega_j}[\beta_\nu \omega_j^{k_\nu}] & \text{for } j = (\mu+1), (\mu+2),\ldots, \end{cases}$$

$$\Delta(\lambda) = \prod_{\nu=1}^{\mu} \rho^{k_\nu} \prod_{j=\mu+1}^{n} e^{\rho\omega_j}([\theta_0] + e^{\rho\omega_\nu}[\theta_1]).$$

We substitute these expressions into formula (34) of §3 in which the last column of the determinant $H(x,\xi,\lambda)$ has been changed in the above way, and distribute the factors of the denominator $\Delta(\lambda)$ as follows. We divide the $(\nu+1)$th row by ρ^{k_ν} for $\nu = 1,\ldots,n$, the μth column by $[\theta_0] + e^{\rho\omega_\mu}[\theta_1]$, and the jth column by $e^{\rho\omega_j}$ for $j = (\mu+1), (\mu+2),\ldots,n$. The formula then takes the following form

$$G(x,\xi,\lambda) = \frac{(-1)^n}{n\rho^{n-1}} \begin{vmatrix} e^{\rho\omega_1 x}[1] & \dfrac{e^{\rho\omega_\mu x}[1]}{[\theta_0] + e^{\rho\omega_\mu}[\theta_1]} & \cdots & e^{\rho\omega_n(x-1)}[1] & P_0 \\ [\alpha_1 \omega_1^{k_1}] & \dfrac{\omega_\mu^{k_1}([\alpha_1] + e^{\rho\omega_\mu}[\beta_1])}{[\theta_0] + e^{\rho\omega_\mu}[\theta_1]} & \cdots & [\beta_1 \omega_n^{k_1}] & P_1 \\ \cdot & \cdot & & \cdot & \\ [\alpha_n \omega_1^{k_n}] & \dfrac{\omega_\mu^{k_n}([\alpha_n] + e^{\rho\omega_\mu}[\beta_n])}{[\theta_0] + e^{\rho\omega_\mu}[\theta_1]} & & [\beta_n \omega_n^{k_n}] & P_n \end{vmatrix}$$

By the conditions (13), all the exponents in this determinant have a real part less than or equal to zero; further, by the results of §4.9 (p. 69), the denominator $[\theta_0] + e^{\rho\omega_\mu}[\theta_1]$ is bounded, in modulus, from below by one and the same number on all the arcs γ_k'. Hence all the elements of the determinant, and therefore the determinant itself, are bounded on these arcs.

Hence on the arcs γ_k' the inequality

$$|G(x,\xi,\lambda)| \leqslant \frac{M}{|\rho|^{n-1}} \tag{20}$$

holds, M being a certain constant.

We now show that the same inequality holds on the arcs γ_k''. To do this we need only to multiply the elements of the first $(\mu - 1)$ columns of the determinant $H(x,\xi,\lambda)$ by $\tfrac{1}{2}z_1(\xi),\tfrac{1}{2}z_2(\xi),\ldots,\tfrac{1}{2}z_{\mu-1}(\xi)$, and those of the μth, $(\mu + 1)$th,...,nth columns by $-\tfrac{1}{2}z_\mu(\xi),-\tfrac{1}{2}z_{\mu+1}(\xi),\ldots,-\tfrac{1}{2}z_n(\xi)$, and add all to the last column. Then by repeating the previous argument, we find that the inequality (20) holds also on the arcs γ_k''.

Hence the inequality (20) has been shown to hold on the part $\gamma_k' + \gamma_k''$ of the arc γ_k lying in S_0. But the preceding arguments are clearly applicable to any domain S_μ. Hence it follows that (20) holds on the part of the arc γ_k lying in S_1 and indeed on the whole arc γ_k. Changing from ρ to λ, we obtain the inequality (12), and the lemma is proved for n odd.

(b). Let n be even; $n = 2\mu$.

This case differs from the preceding one only in that, in two columns, the μth and $(\mu + 1)$th, the numerators

$$e^{\rho\omega_\mu}[1] - [\xi'] \text{ and } e^{\rho\omega_\mu}[1] - [\xi'']$$

appear. These are bounded from below on the arcs γ_k', and therefore the inequality (12) holds on the arcs γ_k'. Similarly, it may be shown that it holds on the arcs γ_k''.

By applying the lemma just proved, we obtain for the integral I_k the estimate

$$|I_k| < \frac{1}{2\pi}\,\frac{M}{R_k^{1-1/n}R_k}\cdot 2\pi R_k = \frac{M}{R_k^{1-1/n}},$$

from which follows immediately the uniform convergence $I_k \to 0$ as $k \to \infty$.

The considerations on p. 84 therefore lead to the following theorem.

THEOREM 2

The Green's function for a differential operator L generated by regular boundary-conditions can be expanded in a uniformly convergent series

$$G(x,s) = -\sum_{\nu=1}^{\infty}\frac{H_\nu(x,\xi)}{\lambda_\nu}, \tag{21}$$

$H_\nu(x,\xi)$ being the residue of the function $G(x,\xi,\lambda)$ at the pole λ_ν.

If all eigenvalues of the operator L are simple zeros of the function $\Delta(\lambda)$, and if the coefficients of the corresponding differential expression l satisfy the differentiability conditions necessary for the existence of the adjoint expression, then (see §3.8)

$$-H_\nu(x,\xi) = y_\nu(x)\overline{z_\nu(\xi)},$$

where $y_\nu(x),z_\nu(x)$ are eigenfunctions of the operators L and L^* corresponding to the eigenvalues λ_ν and $\bar{\lambda}_\nu$ and are normalized so that[1]

$$\int_0^1 y_\nu(x)\overline{z_\nu(x)}dx = 1.$$

---

[1] We shall assume these normalization conditions to be satisfied in Theorems 3 and 4, which follow.

If we apply Theorem 2 to this case, we obtain

THEOREM 3

If all eigenvalues of an operator L of the form considered in §4.1, *which is generated by regular boundary conditions and for which the adjoint differential operator exists, are simple zeros of the function* Δ, *then its Green's function can be expanded in a uniformly convergent series*

$$G(x,\xi) = \sum_{\nu=1}^{\infty} \frac{y_\nu(x)\overline{z_\nu(\xi)}}{\lambda_\nu}. \tag{22}$$

From this theorem we easily obtain one on the representation of a given function.

THEOREM 4

Let L be an operator generated by regular boundary conditions with a differential expression of the form considered in §4.1 *and for which the adjoint differential operator exists. Let all eigenvalues of L be simple zeros of the function* Δ. *Then any function* $f(x)$ *in the domain of definition of the operator L can be expanded in a uniformly convergent series of the eigenfunctions:*

$$f(x) = \sum_{\nu=1}^{\infty} \alpha_\nu y_\nu(x),$$

$$\alpha_\nu = \int_0^1 f(\xi)z_\nu(\xi)d\xi,$$

where $y_\nu(x)$, $z_\nu(x)$ *are the eigenfunctions corresponding to the eigenvalues* λ_ν, $\bar{\lambda}_\nu$ *of the operators* L, L^*.

Proof. We put $Lf = h$; then

$$f(x) = \int_0^1 G(x,\xi)h(\xi)d\xi.$$

In this we substitute for $G(x,\xi)$ its expansion (22). Since the series is uniformly convergent, we may integrate termwise. Hence

$$f(x) = \sum_{\nu=1}^{\infty} \alpha_\nu y_\nu(x),$$

where

$$\alpha_\nu = \frac{1}{\lambda_\nu}\int_0^1 h(\xi)\overline{z_\nu(\xi)}d\xi = \frac{1}{\lambda_\nu}(h,z_\nu) = \frac{1}{\lambda_\nu}(Lf,z_\nu) = \frac{1}{\lambda_\nu}(f,L^*z_\nu) = \frac{1}{\lambda_\nu}(f,\bar{\lambda}_\nu z_\nu)$$

$$= (f,z_\nu).$$

The asymptotic formulae obtained in §4 also enable the convergence to be investigated of an expansion in terms of eigenfunctions of a given function which does not belong to the range of definition of the operator L. The results found are similar to those for ordinary Fourier series; see the papers by Birkhoff [5], Tamarkin [110], and Stone [109a].

Remark. It follows from the Corollary to Theorem 1 in §5.2 that *every* function f belonging to $L^2(a,b)$, and not merely functions f belonging to the domain of the operator, can be expanded as a series in the eigenfunctions of a self-adjoint operator L and this series is convergent in the quadratic mean. Since the eigenfunctions of a self-adjoint operator are orthogonal to one another, the expansion also converges unconditionally.

The question arises whether a similar proposition holds for a non-self-adjoint operator L. G. M. Keselman [49] and V. P. Mikhailov [75] have given an affirmative answer to this question on the assumption that the operator L is generated by regular boundary conditions, and that, for an operator of even order, $\theta_0^2 \neq \theta_{-1}\theta_1$. Their results contain sufficient conditions to ensure that the eigenfunctions of the operator L should form a Riesz basis in the space $L^2(a,b)$ — (see N. B. Bari [3]). We say that a complete system

$$\{\varphi_n(x)\}_{n=1}^{\infty} \subset L^2(a,b)$$

is a *Riesz basis* if for all $f \in L^2(a,b)$

$$\sum_{n=1}^{\infty} \left| \int_a^b f(x)\varphi_n(x)dx \right|^2 < \infty,$$

and if, for any sequence of numbers $c_1, c_2, \ldots$ such that

$$\sum_{n=1}^{\infty} |c_n|^2 < \infty,$$

there exists a function $f \in L^2(a,b)$ such that

$$\int_a^b f(x)\varphi_n(x)dx = c_n\,, \quad n = 1,2,3,\ldots\ .$$

If $\{\varphi_n(x)\}$ is a Riesz basis, then the following lower and upper estimates hold, instead of Parseval's relation:

$$m \int_a^b |f(x)|^2 dx \leqslant \sum_{n=1}^{\infty} \left| \int_a^b f(x)\varphi_n(x)dx \right|^2 \leqslant M \int_a^b |f(x)|^2 dx,$$

where the numbers m and M, $M \geqslant m > 0$, are independent of f.

We remark that operators whose eigenfunctions and associated functions form a Riesz basis are spectral operators: see N. Dunford's survey [21].

4. Expansion in terms of Eigenfunctions when the Boundary Conditions are Separable

The boundary conditions are said to be *separable* if each separate $U_\nu(y)$ contains the values of the function and its derivatives at only one of the two boundary points of the interval. Separable boundary conditions therefore have the form

$$\left.\begin{aligned} U_j(y) &= \sum_{k=0}^{n-1} \alpha_{jk} y_a^{(k)} = 0 \text{ for } j \leqslant m, \\ U_j(y) &= \sum_{k=0}^{n-1} \beta_{jk} y_b^{(k)} = 0 \text{ for } j > m. \end{aligned}\right\} \tag{23}$$

For $m = 0$ or $m = n$, they are equivalent to the Cauchy initial conditions

$$y_a^{(k)} = 0 \text{ or } y_b^{(k)} = 0, \qquad k = 0,1,\ldots,(n-1).$$

It may readily be seen that, if $0 < m < n$, the separable boundary conditions are not regular; the only exception is the case where n is even and $m = \frac{1}{2}n$. In this case the boundary conditions are of the Sturm type and are therefore, under certain conditions, regular (see §4.8).

In the case of separable boundary conditions, the concept of the weight of a matrix is useful.[1] Let A be a matrix of rank m:

$$A = \begin{bmatrix} \alpha_{10} & \cdots & \alpha_{1,n-1} \\ \cdot & & \cdot \\ \cdot & \cdots & \cdot \\ \alpha_{m0} & \cdots & \alpha_{m,n-1} \end{bmatrix};$$

then at least one of the determinants

$$\begin{vmatrix} \alpha_{1j} & \cdots & \alpha_{1j_m} \\ \cdot & \cdots & \cdot \\ \alpha_{mj_1} & & \alpha_{mj_m} \end{vmatrix} \tag{24}$$

does not vanish. We call the sum $j_1 + \ldots + j_m$ *the weight of the determinant* (24), and *the weight of the matrix A* is the greatest weight of its non-vanishing determinants (24).

LEMMA 2

Among the determinants (24) of the matrix A there is just one non-vanishing determinant whose weight is equal to the weight of the matrix A.

---

[1] This concept and also the proof given later in this section are due to M. V. Keldysh [47].

Proof. We prove this assertion by induction. For $m = 1$, the assertion is clearly true.

Suppose it holds when the number of rows in the matrix A is less than m; we shall show that it then holds for a matrix of m rows.

Let the number p be chosen so that the $(p + 1)$th column contains at least one non-zero element, while all the later columns contain only zeros, i.e.

$$A = \begin{bmatrix} \alpha_{10} & & \alpha_{1p} & 0 & & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \alpha_{m0} & & \alpha_{mp} & 0 & \cdots & 0 \end{bmatrix}.$$

Clearly, $p + 1 \geqslant m$.

We consider the determinant

$$\begin{vmatrix} \alpha_{1,p-m+1} & & \alpha_{1p} \\ \cdot & \cdots & \cdot \\ \alpha_{m,p-m+1} & \cdots & \alpha_{mp} \end{vmatrix};$$

if it is not equal to zero, the proposition obviously holds.

Suppose, however, that it vanishes, and let q be the rank of its matrix, so that $q < m$; then $(m - q)$ of its rows are linear combinations of its remaining rows. Without loss of generality we may assume that these are its last $(m - q)$ rows, since an interchange of rows in the matrix A merely changes the signs of the determinants (24). We add to the last $(m - q)$ rows of matrix A certain linear combinations of its other rows (this does not alter the determinants (24)) to bring A into the form

$$A' = \begin{bmatrix} \alpha'_{10} & \cdots & \alpha'_{1,p-m} & \alpha'_{1,p-m+1} & \cdots & \alpha'_{1p} & 0 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \alpha'_{q0} & & \alpha'_{q,p-m} & \alpha'_{q,p-m+1} & \cdots & \alpha'_{qp} & 0 & \cdots & 0 \\ \alpha'_{q+1,0} & & \alpha'_{q+1,p-m} & 0 & \cdots & 0 & 0 & & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \alpha'_{m0} & & \alpha'_{m,p-m} & 0 & & 0 & 0 & & 0 \end{bmatrix}.$$

The rank of the matrix

$$A'_1 = \begin{bmatrix} \alpha'_{1,p-m+1} & & \alpha'_{1,p} \\ \cdot & & \\ \alpha'_{q,p-m+1} & \cdots & \alpha'_{qp} \end{bmatrix}$$

is obviously equal to q, and the rank of the matrix

$$A_2' = \begin{bmatrix} \alpha'_{q+1,0} & \cdots & \alpha'_{q+1,p-m} \\ \cdot & & \cdot \\ \alpha'_{m0} & & \alpha'_{m,p-m} \end{bmatrix}$$

is equal to $(m - q)$. By the hypothesis of the induction, the matrix A_1' has just one non-zero determinant of the maximum weight; let this be

$$D_q = \begin{vmatrix} \alpha'_{1,j_{m-q+1}} & \cdots & \alpha'_{1,j_m} \\ \cdot & & \cdot \\ \alpha'_{q,j_{m-q+1}} & \cdots & \alpha'_{q,j_m} \end{vmatrix}.$$

Similarly, the matrix A_2' has just one non-zero determinant of maximum weight, say

$$D_q' = \begin{vmatrix} \alpha'_{q+1,j_1} & \cdots & \alpha'_{q+1,j_{m-q}} \\ \cdot & \cdots & \cdot \\ \alpha'_{m,j_1} & \cdots & \alpha'_{m,j_{m-q}} \end{vmatrix}.$$

Then the determinant

$$D = \begin{vmatrix} \alpha'_{1,j_1} & & \alpha'_{1,j_{m-q}} & \alpha'_{1,j_{m-q+1}} & \cdots & \alpha'_{1,j_m} \\ \cdot & \cdots & \cdot & & & \cdot \\ \alpha'_{q,j_1} & & \alpha'_{q,j_{m-q}} & \alpha'_{q,j_{m-q+1}} & & \alpha'_{q,j_m} \\ \alpha'_{q+1,j_1} & \cdots & \alpha'_{q+1,j_{m-q}} & 0 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \alpha'_{m,j_1} & & \alpha'_{m,j_{m-q}} & 0 & & 0 \end{vmatrix}$$

is obviously non-zero, and all other non-vanishing determinants of the matrix A' have a smaller weight. For, any such determinant Δ has at least one sub-determinant Δ_q formed out of the first q rows of the matrix A and which, together with its complement Δ_q', is non-zero. The weight of the determinant Δ is equal to the sum of the weights of Δ_q and Δ_q'.

But by definition of D_q and D_q', we have

$$\text{weight of } \Delta_q \leqslant \text{weight of } D_q, \tag{25}$$

$$\text{weight of } \Delta_q' \leqslant \text{weight of } D_q', \tag{26}$$

and the equality signs are possible only if Δ_q coincides with D_q or Δ_q' with D_q'.

Hence

$$\text{weight of } \Delta \leqslant \text{weight of } D,$$

and the equality sign appears only if it occurs in each of (25) and (26). This is clearly only possible if Δ coincides with D.

Hence Lemma 2.

We now consider the matrices

$$A = \begin{bmatrix} \alpha_{10} & \alpha_{1,n-1} \\ \cdot & \cdot \\ \alpha_{m0} & \alpha_{m,n-1} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \beta_{m+1,0} & & \beta_{m+1,n-1} \\ \cdot & & \cdot \\ \beta_{n0} & \cdots & \beta_{n,n-1} \end{bmatrix}$$

formed from the coefficients of the boundary conditions (23).

Since these boundary conditions are linearly independent, the rank of each matrix A and B is equal to the number of its rows; hence we may speak of the weights of these matrices. In future, we shall denote the weight of the matrix A by h and that of the matrix B by l. Further, we shall put $a = 0$ and $b = 1$; as before, this causes no loss of generality.

LEMMA 3 (M. V. Keldysh)

An operator L which is generated by the differential expression $l(y) = y^{(n)}$ and the separable boundary conditions

$$\sum_{k=0}^{n-1} \alpha_{jk} y_0^{(k)} = 0 \qquad \text{for } j \leqslant m,$$

$$\sum_{k=0}^{n-1} \beta_{jk} y_1^{(k)} = 0 \qquad \text{for } j > m,$$

which are not equivalent to the Cauchy initial conditions has infinitely many eigenvalues λ_ν.

For these eigenvalues the following asymptotic formulae hold:

$$\lambda_\nu = \rho_\nu^n, \tag{27}$$

where

$$\rho_\nu = \pi \frac{\nu + \frac{n}{2} - \frac{h+l}{n}}{\sin \frac{2\kappa\pi}{n}} + O\left(\frac{1}{\nu}\right), \tag{28 a}$$

if $n - m = 2\kappa$ and if ν is sufficiently great, and

$$\rho_\nu = \pi e^{-i\pi/n} \frac{\nu + \frac{n}{2} - \frac{h+l}{n}}{\sin \frac{(2\kappa+1)\pi}{n}} + O\left(\frac{1}{\nu}\right) \tag{28 b}$$

if $n - m = 2\kappa + 1$ and if ν is sufficiently great.

Proof. We put $\lambda = \rho^n$; the equation $y^{(n)} = \rho^n y$ has the general integral

$$y = \sum_{j=1}^{n} c_j e^{\rho\omega_j x},$$

where $\omega_j, j = 1,2,\ldots,n$, denote the different nth roots of unity. By substituting this expression in the boundary conditions (23) we obtain a system of homogeneous equations

$$\begin{aligned} &\sum_{q=1}^{n} c_q \sum_{k=0}^{n-1} \alpha_{jk}\omega_q^k\rho^k = 0, && j = 1,2,\ldots,m \\ &\sum_{q=1}^{n} c_q \sum_{k=0}^{n-1} \beta_{jk}\omega_q^k\rho^k e^{\omega_q\rho} = 0, && j = (m+1), (m+2),\ldots,n \end{aligned}$$

for the unknowns $c_1,c_2,\ldots,c_n$. Hence the eigenvalues for the operator L in question are determined by the equation

$$D(\rho) = \begin{vmatrix} \sum\limits_{k=0}^{n-1} \alpha_{1k}\omega_1^k\rho^k & \ldots & \sum\limits_{k=0}^{n-1} \alpha_{1k}\omega_n^k\rho^k \\ \cdot & \ldots & \cdot \\ \sum\limits_{k=0}^{n-1} \alpha_{mk}\omega_1^k\rho^k & \ldots & \sum\limits_{k=0}^{n-1} \alpha_{mk}\omega_n^k\rho^k \\ \sum\limits_{k=0}^{n=1} \beta_{m+1,k}\omega_1^k\rho^k e^{\omega_1\rho} & \ldots & \sum\limits_{k=0}^{n-1} \beta_{m+1,k}\omega_n^k\rho^k e^{\omega_n\rho} \\ \cdot & \ldots & \cdot \\ \sum\limits_{k=0}^{n-1} \beta_{nk}\omega_1^k\rho^k e^{\omega_1\rho} & \ldots & \sum\limits_{k-0}^{n-1} \beta_{nk}\omega_n^k\rho^k e^{\omega_n\rho} \end{vmatrix} = 0 \qquad (29)$$

Clearly, we need discuss only the case $1 \leqslant m \leqslant \frac{1}{2}n$, since the case $\frac{1}{2}n < m < n - 1$ can be reduced to the first case by the substitution $x_1 = 1 - x$.

We put

$$\left.\begin{aligned} A_{ij} &= \sum_{k=0}^{n-1} \alpha_{ik}\omega_j^k\rho^k, \quad i = 1,2,\ldots,m; \quad j = 1,2,\ldots,n \\ B_{ij} &= \sum_{k=0}^{n-1} \beta_{ik}\omega_j^k\rho^k, \quad i = (m+1),\ldots,n; \quad j = 1,2,\ldots,n. \end{aligned}\right\} \qquad (30)$$

By Laplace's expansion theorem,

$$\Delta(\rho) = \sum \pm \begin{vmatrix} A_{1j_1} & \ldots & A_{1j_m} \\ \cdot & & \cdot \\ A_{mj_1} & \ldots & A_{mj_m} \end{vmatrix} \cdot \begin{vmatrix} B_{m+1,j_{m+1}} & \ldots & B_{m+1,j_n} \\ \cdot & & \cdot \\ B_{n,j_{m+1}} & \ldots & B_{n,j_n} \end{vmatrix} . e^{(\omega_{j_{m+1}}+\ldots+\omega_{j_n})\rho}, \qquad (31)$$

the summation being over all possible permutations $j_1, j_2, \ldots, j_m, j_{m+1}, j_{m+2}, \ldots, j_n$ of the numbers $1, 2, \ldots, n$ satisfying $j_1 < j_2 < \ldots < j_n$ and $j_{m+1} < \ldots < j_n$. In other words, $D(\rho)$ has the form

$$D(\rho) = \sum_{k=1}^{N} P_k(\rho) e^{\mu_k \rho}, \tag{32}$$

where $P_k(\rho)$ is a polynomial in ρ of degree $\leqslant n(n-1)$, and all the exponents μ_k are different.

We are concerned with those terms of this sum in which the real part of the expression $\omega_{j_{m+1}} + \ldots + \omega_{j_n}$ takes the greatest value. We shall shortly see that for each such term the polynomial $P_k(\rho)$ associated with the term consists of just a single product of two determinants from (31). Hence it suffices to show in each case that neither of the determinants

$$\begin{vmatrix} A_{1j_1} & \cdots & A_{1j_m} \\ \cdot & & \cdot \\ A_{mj_1} & \cdots & A_{mj_m} \end{vmatrix}, \quad \begin{vmatrix} B_{m+1,j_{m+1}} & & B_{m+1,j_n} \\ \cdot & & \cdot \\ B_{nj_{m+1}} & \cdots & B_{n,j_n} \end{vmatrix} \tag{32 a}$$

in this product vanishes identically. We shall prove this assertion for the first of the determinants; the proof for the second of the determinants is similar. We have:

$$\begin{vmatrix} A_{1j_1} & \cdots & A_{1j_m} \\ \cdot & & \\ A_{mj_1} & \cdots & A_{mj_m} \end{vmatrix} = \sum_{h_1, h_2, \ldots, h_m} \pm \begin{vmatrix} \alpha_{1h_1} & \cdots & \alpha_{1h_m} \\ \cdot & \cdots & \cdot \\ \alpha_{mh_1} & \cdots & \alpha_{mh_m} \end{vmatrix} \begin{vmatrix} \omega_{j_1}^{h_1} & \cdots & \omega_{j_1}^{h_m} \\ \cdot & & \\ \omega_{j_m}^{h_1} & \cdots & \omega_{j_m}^{h_m} \end{vmatrix} \rho^{h_1 + \ldots + h_m} \tag{33}$$

But if $\mathscr{R}(\omega_{j_{m+1}} + \ldots + \omega_{j_n})$ takes the greatest value, then the points $\omega_{j_{m+1}}, \omega_{j_{m+2}}, \ldots, \omega_{j_n}$ on the unit circle are situated as far to the right as they can be in any of their possible arrangements, and consequently, when numbered appropriately, they lie within an angular distance of $2\pi/n$ from one another (Fig. 14). If the numbers $\omega_1, \omega_2, \ldots, \omega_n$ are numbered cyclically in a suitable way, then

$$\omega_{j_k} = \zeta^{k-1} \omega_{j_1}, \qquad k = 1, 2, \ldots, n,$$

where $\zeta = e^{2\pi i/n}$. Introducing the notation $\zeta_k = \zeta^k$, we have

$$\alpha = \begin{vmatrix} \omega_{j_1}^{h_1} & \cdots & \omega_{j_1}^{h_m} \\ \cdot & \cdots & \\ \omega_{j_m}^{h_1} & \cdots & \omega_{j_m}^{h_m} \end{vmatrix} = \begin{vmatrix} 1 & & 1 \\ \zeta_{h_1} & \cdots & \zeta_{h_m} \\ \cdot & \cdots & \\ \zeta_{h_1}^{m-1} & \cdots & \zeta_{h_m}^{m-1} \end{vmatrix} \omega_{j_1}^{h_1 + \cdots + h_m} \neq 0, \tag{34}$$

since the numbers $\zeta_{h_1}, \zeta_{h_2}, \ldots, \zeta_{h_m}$ are all different.

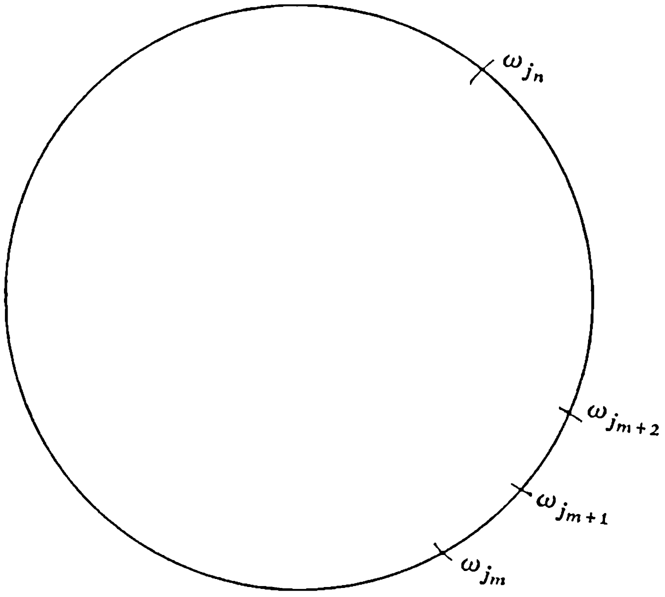

Fig. 14.

Hence we have only now to show that among the various non-zero determinants

$$\begin{vmatrix} \alpha_{1h_1} & \alpha_{1h_m} \\ \cdot & \cdot \\ \alpha_{mh_1} & \alpha_{mh_m} \end{vmatrix} \tag{35}$$

there is just one for which the exponent $h_1 + \ldots + h_m$ of ρ takes the greatest value. But this exponent is equal to the weight of the determinant (35); hence our assertion coincides with the enunciation of Lemma 2. Hence the first, and similarly the second, determinant does not vanish. On the other hand, the values of $\omega_{j_{m+1}} + \ldots + \omega_{j_n}$ for the separate combinations with greatest real part are obviously different. Hence in the corresponding polynomials P_k only one single product of determinants appears, and it has been proved that *the terms which have the greatest value of* $\mathscr{R}(\omega_{j_{m+1}} + \ldots + \omega_{j_n})$ *cannot vanish identically*. For $(n - m)$ even, there are just two such combinations (cf. Fig. 15); for $(n - m)$ odd, just one. In the latter case, we notice that the two combinations having a not smaller real part of $\omega_{j_{m+1}} + \ldots + \omega_{j_n}$ also have non-vanishing coefficients in formula (32), since these too consist of only a single product of determinants.

We examine the cases $(n - m)$ even and $(n - m)$ odd separately.

1. $(n - m)$ *even*, $n - m = 2\kappa$.

By numbering the numbers ω_j suitably we obtain two sums

$$\omega_{-\kappa} + \omega_{-\kappa+1} + \cdots + \omega_{\kappa-1}, \qquad \omega_{-\kappa+1} + \omega_{-\kappa+2} + \cdots + \omega_{\kappa}$$

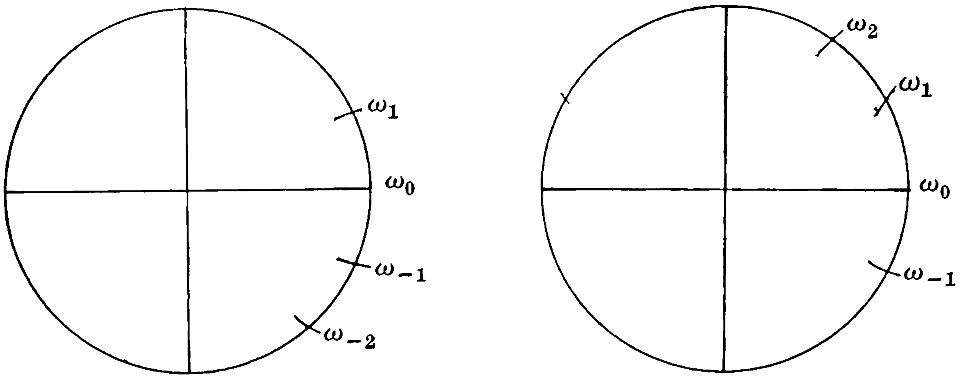

Fig. 15.

with the greatest real part (see Fig. 15, drawn for $\kappa = 2$). We put $\arg \rho = \varphi$ and consider the function $D(\rho)$ in the sector S: $-\frac{\pi}{n} \leqslant \varphi \leqslant \frac{\pi}{n}$. In this sector at least one of the exponents

$$(\omega_{-\kappa} + \omega_{-\kappa+1} + \dots + \omega_{\kappa-1})\rho, \qquad (\omega_{-\kappa+1} + \omega_{-\kappa+2} + \dots + \omega_{\kappa})\rho$$

has a real part greater than the real part of all other exponents in the expression for $D(\rho)$. Let h be the weight of the matrix

$$A = \begin{bmatrix} \alpha_{10} & & \alpha_{1,n-1} \\ \cdot & \dots & \cdot \\ \alpha_{m0} & & \alpha_{m,n-1} \end{bmatrix},$$

and l the weight of the matrix

$$B = \begin{bmatrix} \beta_{m+1,0} & \cdots & \beta_{m+1,n-1} \\ \cdot & \dots & \cdot \\ \beta_{n0} & \cdots & \beta_{n,n-1} \end{bmatrix}.$$

Then by the above investigation, in the sector S we have

$$D(\rho) = e^{(\omega_{-\kappa} + \omega_{-\kappa+1} + \dots + \omega_{\kappa-1})\rho} \alpha' \rho^{h+l} \left[1 + O\left(\frac{1}{\rho}\right)\right] + e^{(\omega_{\kappa+1} + \omega_{-\kappa+2} + \dots + \omega_{\kappa})\rho} \beta' \rho^{h+l} \left[1 + O\left(\frac{1}{\rho}\right)\right].$$

Now α' and β' clearly differ only by a factor of $(-1)^{n-1} e^{2\pi i (h+l)/n}$:

$$\beta' = \alpha' . (-1)^{n-1} e^{2\pi i (h+l)/n},$$

because we obtain the coefficient β' from the coefficient α' by replacing the order $j_1, j_2, \dots, j_n$ in the determinants (32 a) by the order $j_2, j_3, \dots, j_n, j_1$. Hence the sign of the product of determinants in formula (31) changes by $(-1)^{n-1}$, while each of the numbers ω_{j_k} is multiplied by $\zeta = e^{2\pi i/n}$, so that, for example, α in formula (34) is multiplied by $e^{2\pi ih/n}$. Hence we have

$$D(\rho) = \alpha' \rho^{h+l} e^{(\omega_{-\kappa} + \omega_{-\kappa+1} + \dots + \omega_{\kappa-1})\rho} \left\{ 1 + O\left(\frac{1}{\rho}\right) \right. \\ \left. + (-1)^{n-1} e^{2\pi i(h+l)/n} e^{(\omega_\kappa - \omega_\kappa)\rho} \left[1 + O\left(\frac{1}{\rho}\right) \right] \right\}.$$

We put $\omega_\kappa = \sigma + i\tau$; then $\omega_{-\kappa} = \bar{\omega}_\kappa = \sigma - i\tau$, where

$$\tau = \sin \frac{2\kappa\pi}{n}. \tag{36}$$

Hence we have

$$D(\rho) = \alpha' \rho^{h+l} e^{(\omega_{-\kappa} + \dots + \omega_{\kappa-1})\rho} \left\{ 1 + O\left(\frac{1}{\rho}\right) \right. \\ \left. + (-1)^{n-1} e^{2i\tau\rho + 2\pi i(h+l)/n} \left[1 + O\left(\frac{1}{\rho}\right) \right] \right\}, \tag{37}$$

so that the equation $D(\rho) = 0$ is equivalent to the equation

$$(-1)^{n-1} + O\left(\frac{1}{\rho}\right) + e^{2i[\tau\rho + \frac{\pi}{n}(h+l)]} = 0.$$

Hence

$$e^{2i[\tau\rho + \frac{\pi}{n}(h+l)]} = (-1)^n + O\left(\frac{1}{\rho}\right);$$

and so for the required eigenvalues we have

$$2i\left[\tau\rho + \frac{\pi}{n}(h+l)\right] = (2\nu + n)\pi i + O\left(\frac{1}{\rho}\right), \qquad \nu = 0, \pm 1, \pm 2, \dots .$$

Therefore

$$\rho = \pi \frac{\nu + \frac{n}{2} - \frac{h+l}{n}}{\tau} + O\left(\frac{1}{\rho}\right),$$

and so

$$\rho = \pi \frac{\nu + \frac{n}{2} - \frac{h+l}{n}}{\tau} + O\left(\frac{1}{\nu}\right).$$

On the other hand, by applying Rouché's theorem, it may easily be shown that (cf. the detailed investigation in §4.8 for the case of regular boundary conditions), for sufficiently large $|\nu|$, there is in the neighbourhood of the point

$$\pi \frac{\nu + \frac{n}{2} - \frac{h+l}{n}}{\tau} + O\left(\frac{1}{\nu}\right)$$

just one zero ρ_ν of the determinant $D(\rho)$, and

$$\rho_\nu = \pi \frac{\nu + \frac{n}{2} - \frac{h+l}{n}}{\tau} + O\left(\frac{1}{\nu}\right),$$

i.e. by (36),

$$\rho_\nu = \pi \frac{\nu + \frac{n}{2} - \frac{h+l}{n}}{\sin \frac{2\kappa\pi}{n}} + O\left(\frac{1}{\nu}\right).$$

2. $(n - m)$ *odd*; $n - m = 2\kappa + 1$.

In the sector S: $-\frac{2\pi}{n} < \varphi < 0$ the expressions

$$(\omega_{-\kappa} + \omega_{-\kappa+1} + \dots + \omega_\kappa)\rho \text{ and } (\omega_{-\kappa+1} + \omega_{-\kappa+2} + \dots + \omega_{\kappa+1})\rho$$

have a real part greater than any other expression

$$(\omega_{j_{m+1}} + \dots + \omega_{j_n})\rho.$$

By considerations similar to those of case 1 we find

$$D(\rho) = e^{(\omega_{-\kappa} + \omega_{-\kappa+1} + \dots + \omega_\kappa)\rho}\alpha'\left[1 + O\left(\frac{1}{\rho}\right)\right] \\ + (-1)^{n-1}e^{(\omega_{-\kappa+1} + \omega_{-\kappa+2} + \dots + \omega_{\kappa+1})\rho}\alpha' e^{\frac{2\pi i(h+l)}{n}}\left[1 + O\left(\frac{1}{\rho}\right)\right].$$

Hence it follows that $D(\rho)$ has infinitely many zeros ρ_ν in the sector S, where

$$\rho_\nu = \pi e^{-\frac{i\pi}{n}} \frac{\nu + \frac{n}{2} - \frac{h+l}{n}}{\sin \frac{2\kappa+1}{n}\pi} + O\left(\frac{1}{\nu}\right).$$

This concludes the proof of Lemma 3.

We now consider briefly the question of expansion in terms of eigenfunctions and associated functions.

An expansion in terms of the eigenfunctions of a differential operator does not, in the general case, have the usual orthogonal properties. However, as we proved in §5.3, this expansion does have properties similar to those of an ordinary Fourier series expansion if the operator is defined by regular boundary conditions. If the boundary conditions are separable, the situation is more complicated.

W. Ward, [117], investigated the eigenfunction expansion for the boundary-value problem

$$y^{(n)} + \lambda y = 0, \quad y(0) = \ldots = y^{(n-2)}(0) = y(1) = 0. \qquad (38)$$

He proved that an expansion in a uniformly convergent series in the eigenfunctions is possible only for functions which satisfy certain analyticity conditions, and consequently such an expansion is far from being always possible for all functions in the domain of definition of the corresponding operator L.

On these questions of eigenfunction expansions there is quite an extensive literature (see [17], [95], [122], [50a, b], [43], [44], [102], [116]).

Here we shall only go into some results due to A. P. Khromov [50c] and W. Eberhard [24], who make use of Lemma 3 (M. V. Keldysh) in their investigations.

A. P. Khromov considered the operator L corresponding to the differential expression

$$l[y] = y^{(n)} + p_2(x)y^{(n-2)} + \ldots + p_n(x)y, \qquad p_\nu \in C^{(n-\nu)}(0,1) \qquad (39)$$

and general separable boundary conditions. It is not difficult to see that in this case the assertion of Lemma 3 remains true. In what follows we need the concept, introduced by M. K. Fage [26], of an *l-analytical function.* We mean by this a function f to which the differential operator l can be applied arbitrarily often and which satisfies inequalities of the form

$$\left| \left(\frac{d}{dx}\right)^q l^p[f(x)] \right| < C^{pn+q}(pn+q)\,!$$

$$p = 0,1,2,\ldots, \qquad q = 0,1,\ldots,(n-1),$$

(with the exclusion, possibly, of $p = 0$ and $q = 0$) in an arbitrary interval $[\alpha,\beta] \subset (0,1)$; here C is a constant which depends only on l, f, α and β. It turns out that the properties of an l-analytical function are in many respects similar to those of an ordinary analytical function.

Let $\{\varphi_k(x)\}_{k=1}^{\infty}$ be the sequence of all the eigenfunctions and associated functions of the operator L under consideration, numbered in order of increasing absolute values of the eigenvalues. A. P. Khromov has established the following proposition.

Theorem 5.

If the series $\sum\limits_{k=1}^{\infty} a_k\varphi_k(x)$ *converges uniformly on some interval* $[x_0,x_1] \subset (0,1)$, *then*:

(*a*) *the several series*

$$\sum_{k=1}^{\infty} q_k \left(\frac{d}{dx}\right)^q l^p[\varphi_k(x)], \qquad p = 0,1,2,\dots, \qquad q = 0,1,\dots,(n-1)$$

converge absolutely and uniformly on every interval $[0,a] \subset (0,x_1)$.

(*b*) *the sum* $f(x)$ *of the series*

$$\sum_{k=1}^{\infty} a_k \varphi_k(x)$$

is an l-analytical function;

(*c*) *the functions*

$$f(x), \quad l[f(x)], \quad l^2[f(x)],\dots$$

satisfy the boundary conditions at $x = 0$.

Conversely, if a function $f(x)$ *which is summable on the interval* $(0,1)$ *is an l-analytical function in the interval* $[0,a)$, $0 < a < 1$, *and if the functions* $f(x)$, $l[f(x)]$, $l^2[f(x)],\dots$ *satisfy the boundary conditions at* $x = 0$, *then* $f(x)$ *can be expanded in the system* $\{\varphi_k(x)\}_{k=1}^{\infty}$ *as a series which converges uniformly in every interval* $[0,b] \subset [0,a) \cap [0,R)$, *where the number R is defined by*

$$\frac{1}{R} = \overline{\lim_{k\to\infty}} \left\{ \frac{1}{k!} \left| \left(\frac{d}{dx}\right)^{k-n[k/n]} l^{[k/n]}[f(x)] \right|_{x=0} \right\}^{1/k}$$

$[x]$ *being the integral part of the number x.*

W. Eberhard's results ([28] – [31]) have much in common with those of Khromov. Eberhard considered an operator L corresponding to the differential expression (39) with coefficients $p_\nu(x)$ which are holomorphic in the circle $|x| \leq 1$ in the complex x-plane, and to general separable boundary-conditions. He proved that, if the series $\sum_{k=1}^{\infty} a_k\varphi_k(x)$ converges uniformly on the interval $[x_1,x_2]$, then this series converges uniformly in every closed subdomain in an open polygon (and consequently that the sum of the series is regular in this polygon) whose vertices are at the points

$$x_1 \cdot \frac{\cos 2\kappa\pi/n}{\cos \pi/n} \cdot e^{(2\nu-1)i\pi/n}, \qquad \nu = 1,\dots,n,$$

in the case where $n - m = 2$,
and at the points

$$x_1 \cdot \frac{\cos (2\kappa + 1)\pi/n}{\cos \pi/n} \cdot e^{2\nu i\pi/n}, \qquad \nu = 1,\dots,n,$$

in the case where $n - m = 2\kappa + 1$;

here, as usual, m is the number of boundary conditions at $x = 0$. Eberhard has also obtained certain necessary conditions, in terms of ordinary analyticity, for the function $f(x)$ to admit a uniformly convergent expansion as a series in the system $\{\varphi_k(x)\}_{k=1}^{\infty}$; and here certain additional restrictions, as well as holomorphism, are imposed on the coefficients of the differential expression (39).

5. The Case of Multiple Poles of the Green's Function; *m*-fold Completeness; Keldysh's Theorem

We again consider, as we did earlier, the generalized eigenvalue problem

$$l(y) = f, \tag{60}$$

$$U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n; \tag{61}$$

let $\lambda_1,\lambda_2,\lambda_3,\ldots$ be all the eigenvalues of this problem. They may in general be multiple zeros of the characteristic determinant $\Delta(\lambda)$ and consequently multiple poles of the Green's function. Let λ be any one of these eigenvalues, and let the eigenfunction and its associated functions which occur in the canonical system belonging to λ be

$$y,y_1,\ldots,y_{q-1}.$$

We construct m systems of functions

$$y^{(\nu,k)} = \left[\frac{d^\nu}{dx^\nu} e^{\lambda t}\left(y_k + y_{k-1}\frac{t}{1!} + \ldots + y\frac{t_k}{k!}\right)\right]_{t=0}, \quad \nu = 0,1,2,\ldots,(m-1).$$

Each of these systems

$$y^{(\nu,0)},y^{(\nu,1)},\ldots,y^{(\nu,q-1)}$$

will be called a *derived system* from the original system $y,y_1,\ldots,y_{q-1}$.

The system of all eigenfunctions and associated functions of the boundary-value problem (60), (61) is said to be *m-fold complete* if every system of m functions

$$\{f_1(x),f_2(x),\ldots,f_m(x)\},$$

each of which has continuous derivatives up to order n inclusive and satisfies the boundary conditions (61), can be represented as the limits of m uniformly convergent sequences of finite linear combinations

$$f_\nu(x) = \lim_{N\to\infty} \sum_{i,k} C_N^{(i)} y_i^{(\nu,k)}$$

whose coefficients are independent of ν.

The considerations in §5.3, §5.4, together with certain refined results in the theory of analytic functions, and a formula for the principal part of the Green's function (see §3.9), enable the following general theorem to be derived, which we merely state here without proof.

THEOREM 6. (M. V. Keldysh, [47])

Let

$$l(y) = y^{(n)} + p_1(x,\lambda)y^{(n-1)} + \dots + [p_n(x,\lambda) + \lambda^m]y,$$

where $p_k(x,\lambda)$ *is a polynomial in* λ *of degree* $< \frac{km}{n}$; *and suppose the boundary conditions may take the form*

$$\sum_{k=1}^{n} \alpha_{jk} y_a^{(k-1)} = 0, \qquad j = 1,2,\dots,p; \qquad 0 < p < n,$$

$$\sum_{k=1}^{n} \beta_{jk} y_b^{(k-1)} = 0, \qquad j = (p+1),(p+2),\dots,n,$$

or

$$y_a^{(k)} = y_b^{(k)}, \qquad k = 0,1,2,\dots,(n-1).$$

Then the system of eigenfunctions and associated functions for this boundary-value problem is m-fold complete.

In the article by V. N. Vizitei and A. S. Markus [114] it is proved that Theorem 6 remains valid also in the case of regular boundary conditions, where, when n is even, it is required that $\theta_0^2 \neq \theta_{-1}\theta_1$. Further, it remains true in the case of general self-adjoint boundary-conditions; the conditions of periodic type, which feature in Theorem 6, are a particular case of these last.

An expanded exposition of M. V. Keldysh's results (in particular, those on which Theorem 6 is based) is given in the monograph by I. Ts. Gokhberg and M. G. Krein [33]. In this monograph have been collected the most recent results in the spectral theory of completely continuous non-selfadjoint operators. (We remark that the resolvent, which we have discussed above, of the operator L is a completely continuous operator.)

In addition to the problem of eigenfunction expansions, a problem of great interest is that of determining a differential operator from the specification of its spectral characteristics, e.g., from the eigenvalues and norms of the eigenfunctions (with known initial values). Particular aspects of this problem are considered in V. M. Levitan's book [63f] and in the articles by Z. L. Leibenson [59a, b].

Surveys of the general state of the theory of linear non-selfadjoint operators, both differential and those specified abstractly, are given in the papers [21], [81i], [48], [73].

The articles [52], [72a], [100], [23b], [59a] may also be consulted on the subjects discussed in this chapter.

CHAPTER III

DIFFERENTIAL OPERATORS IN A SPACE OF VECTOR-FUNCTIONS

§6. *Basic Concepts*

1. Linear Differential Expressions in a Space of Vector-Functions

Let R_m denote an m-dimensional complex vector space; i.e. R_m consists of all vectors

$$y = (y_1, y_2, \ldots, y_m),$$

where each y_r is a complex number.

Functions $y = y(x)$, of the real independent variable x, whose values are not numbers but vectors in R_m are called *vector-functions*. A vector-function is therefore simply a system of m complex-valued functions

$$y(x) = (y_1(x), y_2(x), \ldots, y_m(x));$$

each of the scalar-functions $y_r(x)$ is called *a component of the vector-function* $y(x)$.

The function $y(x)$ is said to be *continuous* at the point x_0 if all its components are continuous at x_0.

Similarly, a function $y(x)$ is said to be *differentiable* if each of its components is differentiable, and by definition

$$y'(x) = (y_1'(x), y_2'(x), \ldots, y_m'(x)).$$

Derivatives of higher order are defined in a similar way.

It may easily be seen that:

$$(y + z)' = y' + z', \qquad (\lambda y)' = \lambda' y + \lambda y',$$

$$(y, z)' = (y', z) + (y, z')$$

where

$$(y, z) = \sum_{k=1}^{m} y_k(x)\overline{z_k(x)}.$$

In addition to vector-functions we shall also be concerned with *operator-functions*. The values of operator-functions are linear[1] operators in R_m. Such operators can be represented by means of square matrices $A(x) = [a_{jn}(x)]$ of order m, whose elements are scalar functions. We shall not, from now on, make any distinction between operators and the matrices which represent them.

---

[1] The reader is reminded that the author's definition of linearity, on p. 2, includes homogeneity.

Essentially, operator-functions are also vector-functions, since the aggregate of all linear operators is a vector-space of dimension m^2. Consequently an operator-function $A(x)$ will be said to be *continuous at the point* x_0 if all its functions $a_{jk}(x)$ are continuous at x_0, and to be *differentiable at* x_0 if all the functions $a_{jk}(x)$ are differentiable at x_0. So, by definition, $A'(x)$ is the matrix whose elements are $a'_{jk}(x)$. We see without more ado that the following rules hold:

$$(A + B)' = A' + B', \qquad (\lambda A)' = \lambda' A + \lambda A',$$

$$(AB)' = A'B + AB', \qquad (Ay)' = A'y + Ay'.$$

We denote by $C^{(n)}$ the aggregate of all vector-functions $y(x)$ which have continuous derivatives up to the nth order inclusive in a fixed interval $[a,b]$. Let $P_0(x),P_1(x),\ldots,P_n(x)$ be operator-functions which are continuous in $[a,b]$, and suppose $\det P_0(x) \neq 0$ in $[a,b]$. An expression of the form[1]

$$l(y) = P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_n(x)y \tag{1}$$

is called *a linear differential expression in the space of vector-functions*.

It is defined for an arbitrary vector-function $y(x)$ belonging to $C^{(n)}$, and itself represents a vector-function, continuous in the interval $[a,b]$. The theory developed below for such differential expressions and the differential operators corresponding to them is formally similar to the theory already presented for scalar functions.

We remark that, essentially, $l(y)$ is a system of m differential expressions of the nth order which depend on m scalar functions $y_1(x),y_2(x),\ldots,y_m(x)$. For $n = 1$, we consider, for example, the expression

$$l(y) = y' + P(x)y,$$

and resolve the vector-function $z = l(y)$ into its components by putting

$$l(y) = (l_1(y),l_2(y),\ldots,l_m(y)).$$

Then we have

$$l_j(y) = y'_j + \sum_{k=1}^{m} P_{jk}(x)y_k, \qquad j = 1,2,\ldots,m,$$

where $P_{jk}(x)$ are the elements of the matrix $P(x)$.

---

[1] We remind the reader that, e.g., the expression $A(x)y^{(n-1)}$ denotes the result of applying the operator $A(x)$ to the vector $y^{(n-1)}(x)$.

2. Boundary Conditions

We denote by

$$y_a, y'_a, \ldots, y_a^{(n-1)};\ y_b, y'_b, \ldots, y_b^{(n-1)}$$

the value of the vector-function and its first (n — 1) derivatives at the points a and b respectively, so that $y_a, \ldots, y_b^{(n-1)}$ are vectors in the space R_m. We put

$$U(y) = A_0 y_a + A_1 y'_a + \ldots + A_{n-1} y_0^{(n-1)} \\ + B_0 y_b + B_1 y'_b + \ldots + B_{n-1} y_b^{(n-1)}, \quad (2)$$

where $A_0, \ldots, A_{n-1}, B_0, \ldots, B_{n-1}$ are fixed linear operators in the space R_m. If several such forms are given, $U_1(y), U_2(y), \ldots, U_q(y)$, then equations of the form

$$U_1(y) = 0, \quad U_2(y) = 0, \ldots, U_q(y) = 0 \quad (3)$$

are called *boundary conditions.*

We denote by $\mathscr{D}$ the aggregate of all functions $y \in C^{(n)}$ which satisfy the conditions (3). Let L be an operator, with $\mathscr{D}$ as its domain of definition, which is defined by the equation

$$Ly = l(y);$$

then L is called the *differential operator which is generated by the differential expression* $l(y)$ *and the boundary conditions* (3).

We put

$$U_\nu(y) = A_{\nu,0} y_a + \ldots + A_{\nu,n-1} y_a^{(n-1)} + B_{\nu,0} y_b + \ldots + B_{\nu,n-1} y_b^{(n-1)}, \\ \nu = 1, 2, \ldots, q.$$

We will assume in the definition of $\mathscr{D}$ that the forms $U_\nu(y)$ are linearly independent; this implies that the rank of the matrix formed from all the elements of the matrices $[A_{\nu j}]$, $[B_{\nu j}]$, viz.

$$\begin{bmatrix} A_{10}, \ldots, A_{1,n-1}, B_{10}, \ldots, B_{1,n-1} \\ A_{20}, \ldots, A_{2,n-1}, B_{20}, \ldots, B_{2,n-1} \\ \cdot \quad \cdots \cdot \qquad \cdots \cdot \\ A_{q0}, \ldots, A_{q,n-1}, B_{q0}, \ldots, B_{q,n-1} \end{bmatrix}$$

is equal to mq; for each form $U_\nu(y)$ has m components.

From now on we shall mainly be concerned with the case $q = n$.

3. The Homogeneous Operator Equation

We want now to examine the following homogeneous equation in an unknown operator function $Y(x)$:

$$l(Y) = P_0(x) Y^{(n)} + P_1(x) Y^{(n-1)} + \ldots + P_n(x) Y = 0. \quad (4)$$

As before, it is assumed that $P_0(x),\ldots,P_n(x)$ are continuous and that $\det P_0 \neq 0$ in the fixed interval $[a,b]$.

We mention some of the most elementary properties of solutions of this equation; they resemble the properties of solutions of ordinary, homogeneous, nth-order equations.

Solutions $Y_1, Y_2,\ldots, Y_n$ of (4) are said to be *linearly independent* if the equation

$$Y_1C_1 + Y_2C_2 + \ldots + Y_nC_n = 0,$$

where $C_1,C_2,\ldots,C_n$ are constant operators, holds only for $C_1 = C_2 = \ldots = C_n = 0$; otherwise they are *linearly dependent.*

A system of n solutions $Y_1, Y_2,\ldots, Y_n$ *is linearly independent if and only if the determinant of the* $n^2 \times n^2$ *matrix*

$$\begin{bmatrix} Y_1 & Y_2 & & Y_n \\ Y_1' & Y_2' & & Y_n' \\ \cdot & & & \\ Y_1^{(n-1)} & Y_2^{(n-1)} & \ldots & Y_n^{(n-1)} \end{bmatrix}$$

formed from the matrices $Y_j^{(\nu)}$ *is non-zero.*

If the solutions $Y_1, Y_2,\ldots, Y_n$ *are linearly independent, then any other solution* Y *has the form*

$$Y = Y_1C_1 + Y_2C_2 + \ldots + Y_nC_n. \tag{5}$$

Consequently, *formula* (5) *defines the general integral of equation* (4).

If Y is a solution of equation (4), and if c is a fixed, constant vector, then we get a solution of the equation $l(y) = 0$ on putting $y(x) = Y(x)c$. To see this, we need only apply both sides of equation (4) to the vector c.

Hence, and from formula (5), we can at once infer:

Any solution of the equation $l(y) = 0$ *has the form*

$$y = Y_1c_1 + Y_2c_2 + \ldots + Y_nc_n, \tag{6}$$

where $c_1,c_2,\ldots,c_n$ *are arbitrary constant vectors in* R_m.

4. The Homogeneous Boundary-Value Problem

The problem of determining a vector-function y which shall satisfy the conditions

$$l(y) = 0, \tag{7}$$

$$U_\nu(y) = 0, \qquad \nu = 1,2,\ldots,n, \tag{8}$$

is called the *homogeneous boundary-value problem.*

We consider the $n^2 \times n^2$ matrix

$$U = \begin{bmatrix} U_1(Y_1) & U_1(Y_2) & \cdots & U_1(Y_n) \\ U_2(Y_1) & U_2(Y_2) & \cdots & U_2(Y_n) \\ & \cdot & \cdots & \\ U_n(Y_1) & U_n(Y_2) & & U_n(Y_n) \end{bmatrix} \tag{9}$$

The results in §6.3 imply:

I. *A homogeneous boundary-value problem* (7), (8) *has a non-trivial solution if and only if the determinant of the matrix U vanishes.*

5. Lagrange's Formula; the Adjoint Differential Expression

In addition to the assumptions made in §6.1, we now further require that, for $k = 0,1,2,\ldots,n$, the coefficient matrices $P_k(x)$ shall each be continuously differentiable $(n - k)$ times. We denote the scalar product of the vectors $y, z \in R_m$ by (y,z). Integrating by parts, we obtain

$$\int_a^b (l(y),z)dx = P(\eta,\zeta) + \int_a^b (y,l^*(z))dx, \tag{10}$$

where $P(\eta,\zeta)$ is a bilinear form in

$$\eta = (y_a,y_a',\ldots,y_a^{(n-1)},y_b,y_b',\ldots,y_b^{(n-1)})$$

and

$$\zeta = (z_a,z_a',\ldots,z_a^{(n-1)},z_b,z_b',\ldots,z_b^{(n-1)}),$$

and where[1]

$$l^*(z) = (-1)^n(P_0^*z)^{(n)} + (-1)^{n-1}(P_1^*z)^{(n-1)} + \ldots + P_n^*z. \tag{11}$$

The differential expression $l^*(z)$ is said to be *adjoint* to $l(z)$. Formula (10) will be referred to as *Lagrange's formula.* A differential expression $l(y)$ is said to be *self-adjoint* if $l^*(y) = l(y)$.

Repeating the argument in §1.5, the following general representation for a self-adjoint differential expression is obtained:

II. *Any self-adjoint differential expression is a sum of self-adjoint differential expressions of the form*

$$l_{2\nu}(y) = (Py^{(\nu)})^{(\nu)},\ l_{2\nu-1}(y) = \tfrac{1}{2}[(iPy^{(\nu-1)})^{(\nu)} + (iPy^{(\nu)})^{(\nu-1)}],$$

where $P = P(x)$ denotes an operator function all of whose values are Hermitian matrices in R_n.

---

[1] If $A = [a_{pq}]$ is a square matrix of the nth order, we denote by A^* the *adjoint matrix*, i.e. the matrix $B = [b_{pq}]$ for which $b_{pq} = \bar{a}_{qp}$. A matrix is called *Hermitian* if $A^* = A$.

6. Adjoint Boundary Conditions; Adjoint Operators

We supplement any given set of linearly independent forms $U_1,\dots,U_n$ to form a complete system of linearly independent forms $U_1,U_2,\dots,U_{2n}$. Repeating the argument of §1.6, we can transform Lagrange's formula to

$$\int_a^b (l(y),z)dx = (U_1,V_{2n}) + (U_2,V_{2n-1}) + \dots + (U_{2n},V_1) + \int_a^b (y,l^*(z))dx, \quad (12)$$

where $V_1,V_2,\dots,V_{2n}$ are linearly independent forms in the variables

$$z_a,z_a',\dots,z_a^{(n-1)},z_b,z_b',\dots,z_b^{(n-1)}.$$

The boundary conditions

$$V_\nu = 0, \qquad \nu = 1,2,\dots,n, \quad (13)$$

(or any conditions equivalent to them) are said to be *adjoint* to the boundary conditions

$$U_\nu = 0, \qquad \nu = 1,2,\dots,n. \quad (14)$$

The operator generated by the differential expression $l^*(y)$ and the boundary conditions (13) is said to be *adjoint* to the operator L generated by the differential expression $l(y)$ and the boundary conditions (14). It will be denoted by L^*.

It follows from formula (12) that, for the operators L and L^*, the equation

$$\int_a^b (Ly,z)dx = \int_a^b (y,L^*z)dx \quad (15)$$

holds.

Conversely, if a differential operator L_1 satisfies the condition

$$\int_a^b (Ly,z)dx = \int_a^b (y,L_1z)dx$$

for all functions y,z in the domains of definition of L and L_1 respectively, then $L_1 = L^*$.

For vector-functions $y(x)$ we define the *scalar product* $\langle y,z\rangle$ by

$$\langle y,z\rangle = \int_a^b (y(x),z(x))dx.$$

Then formula (15) can also be written as

$$\langle Ly,z\rangle = \langle y,L^*z\rangle. \quad (16)$$

An operator L is *self-adjoint* if $L^* = L$. In other words, an operator L is self-adjoint if it is generated by a self-adjoint differential expression and self-adjoint boundary conditions. For a self-adjoint operator L, equation (16) takes the form

$$\langle Ly,z\rangle = \langle y,Lz\rangle.$$

The homogeneous boundary-value problem $L^*z = 0$ is said to be *adjoint* to the homogeneous boundary-value $Ly = 0$. By an argument essentially the same as that in §1.7 we obtain:

The rank of a homogeneous boundary-value problem is the same as that of the adjoint homogeneous boundary-value problem.

In particular,

A homogeneous boundary-value problem has a non-trivial solution if and only if the adjoint homogeneous boundary-value problem has a non-trivial solution.

7. Eigenvalues and Eigenfunctions of a Differential Operator

A number λ is called an *eigenvalue* of an operator L if there is in the domain of definition of L a vector-function $y \not\equiv 0$ such that

$$Ly = \lambda y.$$

All the basic theorems in §2.1, 2.4, 2.5 remain true for operators in the space of vector-functions. In particular, the eigenvalues are the zeros of the characteristic determinant

$$\Delta(\lambda) = \begin{vmatrix} U_1(Y_1) & \dots & U_1(Y_n) \\ \cdot & & \cdot \\ U_n(Y_1) & & U_n(Y_n) \end{vmatrix}, \tag{17}$$

where $Y_1, Y_2, \dots, Y_n$ are linearly independent solutions of the operator equation $l(Y) - \lambda Y = 0$.

These solutions can be chosen to be integral, analytic, operator functions of the parameter λ; hence $\Delta(\lambda)$ also is an integral, analytic function, and Theorem I of §2.1 carries over to the present case.

The argument in §2.2 also applies here, so far as the generalized eigenvalue problem is concerned.

We may observe that the ordinary eigenvalue problem in the space of vector-functions is equivalent to a certain generalized eigenvalue problem for scalar functions. To illustrate this, we will examine the eigenvalue problem

$$\frac{dy}{dx} + A(x)y = \lambda y, \qquad 0 \leqslant x \leqslant 1, \tag{18}$$

$$y(0) = y(1), \tag{19}$$

where y is a vector-function in the 2-dimensional space $R_2 = y(x) = \{y_1(x), y_2(x)\}$, and $A(x)$ is the operator defined in this space by the matrix

$$A(x) = \begin{bmatrix} 0 & 1 \\ p(x) & 0 \end{bmatrix}.$$

Consequently

$$A(x)y = \{y_2(x), p(x)y_1(x)\},$$

and written in terms of coordinates the problem (18), (19) takes the form

$$\left.\begin{aligned} \frac{dy_1}{dx} + y_2(x) &= \lambda y_1(x), \\ \frac{dy_2}{dx} + p(x)y_1(x) &= \lambda y_2(x), \end{aligned}\right\} \tag{20}$$

$$y_1(0) = y_1(1); \quad y_2(0) = y_2(1). \tag{21}$$

By differentiating the first equation in (20) to obtain

$$\frac{d^2y_1}{dx^2} + \frac{dy_2}{dx} = \lambda \frac{dy_1}{dx},$$

and then using this equation with (20) to eliminate y_2 and $\frac{dy_2}{dx}$, we obtain for y_1 the differential equation

$$\frac{d^2y_1}{dx^2} - 2\lambda \frac{dy_1}{dx} + [\lambda^2 - p(x)]y_1 = 0. \tag{22}$$

By the first equation in (20) again, the boundary conditions may be written in the form:

$$\left.\begin{aligned} & y_1(0) = y_1(1), \\ & \left[\frac{dy_1}{dx} - \lambda y_1\right]_{x=0} = \left[\frac{dy_1}{dx} - \lambda y_1\right]_{x=1}. \end{aligned}\right\} \tag{23}$$

The original problem therefore reduces to the following generalized eigenvalue problem:

$$\frac{d^2y_1}{dx^2} - 2\lambda \frac{dy_1}{dx} + [\lambda^2 - p(x)]y_1 = 0,$$

$$y_1(0) = y_1(1); \ \left[\frac{dy_1}{dx} - \lambda y_1\right]_{x=0} = \left[\frac{dy_1}{dx} - \lambda y_1\right]_{x=1}.$$

This remark often enables a generalized eigenvalue problem in the space of scalar or vector-functions to be reduced to a certain ordinary eigenvalue problem in the space of vector-functions.

In connection with the generalized eigenvalue problem in the space of vector-functions, see also [95a,b, 122].

8. Differential Operators of the First Order

The eigenvalues and eigenfunctions are determined most easily in the case of a differential operator L of the first order. The operator is generated by the differential expression

$$l(y) = y' + P(x)y \tag{24}$$

and the boundary conditions

$$Ay_a + By_b = 0. \tag{25}$$

We want to find the eigenvalues and eigenfunctions of the operator L. To do this, we first determine the solution of the equation

$$y' + P(x)y = \lambda y. \tag{26}$$

Putting $y = e^{\lambda x}z$ in (26), we see that z has to satisfy the equation

$$z' + P(x)z = 0. \tag{27}$$

To each vector $c \in R_m$ we assign that solution $z(x)$ of (27) which satisfies the initial condition $[z]_{x=a} = c$. Since the solution is unique, the correspondence

$$c \rightarrow z(x)$$

is one-to-one. Moreover, it is obviously linear: $c_1 \rightarrow z_1(x)$ and $c_2 \rightarrow z_2(x)$ imply $\lambda c_1 + \mu c_2 \rightarrow \lambda z_1(x) + \mu z_2(x)$ for arbitrary numbers λ,μ. Hence for each fixed x, this correspondence defines a linear operator $Q(x)$:

$$z(x) = Q(x)c;$$

and, by the definition of $Q(x)$, $Q(a)c = c$.

Hence the general solution of equation (26) is of the form

$$y = e^{\lambda x}Q(x)c, \tag{28}$$

where c is an arbitrary vector in R_m.

Substituting this expression in the boundary conditions (25) and removing a factor $e^{\lambda a}$, we obtain

$$[A + e^{\lambda(b-a)}BQ(b)]c = 0. \tag{29}$$

Next we consider the generalized eigenvalue problem

$$[A + \mu BQ(b)]c = 0 \tag{30}$$

for the operators A and $BQ(b)$ in the finite-dimensional space R_m. Let $\mu_1,\mu_2,\ldots,\mu_r$ be all the eigenvalues, and $c_1,c_2,\ldots,c_r$ the corresponding linearly independent eigenvectors for this problem. Equation (29) shows that $e^{\lambda(b-a)}$ is an eigenvalue and c the corresponding eigenvector of problem (30).

Hence *all the eigenvalues of the operator L are of the form*

$$\lambda = \frac{1}{b-a}\ln\mu_j + 2k\pi i, \qquad j = 1,2,\dots,r : k = 0,\pm1,\pm2,\pm3,\dots;$$

and the corresponding eigenfunctions are

$$\left[\exp\left(\frac{1}{b-a}\ln\mu_j + 2k\pi i\right)x\right] Q(x)cj.$$

Many authors (see, e.g. [6]) have investigated the more general eigenvalue-problem

$$\frac{dy}{dx} + P(x)y = \lambda Q(x)y, \tag{31}$$

$$Ay_a + By_b = 0, \tag{32}$$

with operator functions $P(x), Q(x)$ continuous in $[a,b]$.

The corresponding results are formulated most simply for the case where, for each x in the interval $[a,b]$, all the roots of the characteristic equation of the operator $Q(x)$ are different, say, $\mu_1,\mu_2,\dots,\mu_m$. In this case, they can be numbered so that for each x the functions $\mu_1(x),\mu_2(x),\dots,\mu_m(x)$ are continuous. Hence there exists a non-singular matrix S (i.e. one for which det $S(x) \neq 0$) such that

$$S^{-1}(x)Q(x)S(x) = M(x),$$

where $M(x)$ is a diagonal matrix having $\mu_1(x),\mu_2(x),\dots,\mu_m(x)$ as its diagonal elements.

We assume that $Q(x)$ is a differentiable function in the interval $[a,b]$. Then $S(x)$ and $M(x)$ are also differentiable. Using the substitution $y = S(x)\hat{y}$, we can reduce equation (31) to the form

$$\frac{d\hat{y}}{dx} + \hat{P}(x)\hat{y} = \lambda M(x)\hat{y},$$

where

$$\hat{P}(x) = S^{-1}(x)S'(x) + S^{-1}(x)P(x)S(x).$$

Henceforward, therefore, $Q(x)$ may be taken to be a diagonal matrix. It turns out that for large $|\lambda|$ the solution of (31) behaves approximately like that of

$$\frac{dy}{dx} = \lambda Q(x)y.$$

More precisely: there is a fundamental matrix $Y(x)$ of solutions of equation (31) for which

$$Y(x) = \left[Y_0(x) + O\left(\frac{1}{\lambda}\right)\right] \exp\left(\lambda\int_a^x Q(t)dt\right),$$

where $Y_0(x)$ is a definite matrix independent of λ.

This result enables asymptotic formulae for the eigenvalues and eigenfunctions to be obtained, and also a theorem to be derived on expansions in terms of eigenfunctions of a boundary-value problem, exactly similar to the expansions in §§4 and 5. For more details, see [6].

§7. *Green's Function for a Differential Operator*

1. Inversion of a Differential Operator

Let L be a differential operator for which the boundary-value problem $Ly = 0$ has merely the trivial solution. Then L has an inverse operator L^{-1}. We shall show that L^{-1} is an integral operator, the kernel of which is represented by an operator function in two variables. This function is called the *Green's function* for the operator.

The precise definition of Green's function for the present case is similar to that given in §3.3 and now runs:

An operator function $G(x,\xi)$ in the variables x,ξ is called the *Green's function* for the operator L if it satisfies the conditions:

1°. *Each value of the function* $G(x,\xi)$ *is a linear operator in the space* R_m.

2°. *The function* $G(x,\xi)$ *is continuous and has continuous derivatives with respect to* x *up to the* $(n-2)$*th order inclusive in the square* $a \leqslant x,\xi \leqslant b$.

3°. *For each fixed value of* ξ *in the interval* (a,b), *the function* $G(x,\xi)$ *has continuous derivatives with respect to* x *of the* $(n-1)$*th and nth orders respectively in each of the intervals* $[a,\xi)$ *and* $(\xi,b]$. *The derivative of the* $(n-1)$*th order makes a jump in value,* $[P_0(\xi)]^{-1}$, *at* $x = \xi$: *i.e.*

$$\frac{\partial^{n-1}}{\partial x^{n-1}} G(\xi + 0,\xi) - \frac{\partial^{n-1}}{\partial x^{n-1}} G(\xi - 0,\xi) = [P_0(\xi)]^{-1}.$$

4°. *For a fixed* ξ *in the interval* (a,b), *the function* $G(x,\xi)$, *regarded as a function of* x, *satisfies the boundary conditions* $U_\nu(G) = 0, \nu = 1,2,\ldots,n$, *and, in each of the intervals* $[a,\xi)$ *and* $(\xi,b]$, *the operator equation* $l(G) = 0$, $l(y)$ *and* $U_\nu(y) = 0$ *being the differential expression and boundary conditions which generate the operator* L.

Repeating the arguments of §3.3–3.5, we obtain the following results:

THEOREM 1

If the boundary-value problem $Ly = 0$ *has only the trivial solution, then the operator* L *has just one Green's function.*

THEOREM 2

If the boundary-value problem $Ly = 0$ *has only the trivial solution, then the* *equation $Ly = f$, where f is any continuous vector function $f(x)$, has just one solution, which is given by the formula*[1]

$$y(x) = \int_a^b G(x,\xi)f(\xi)d\xi. \tag{1}$$

Theorem 2 means that L^{-1} is an integral operator with the kernel $G(x,\xi)$.

The kernel $G^*(x,\xi) = [G(\xi,x)]^*$ is called the *adjoint* of the kernel $G(x,\xi)$. A kernel is said to be *Hermitian* if it is the same as its adjoint: $G(x,\xi) = [G(\xi,x)].^*$ Theorems III and IV of §3.5 and all the theorems in §3.6 remain valid.

2. Green's Function for the Operator $L - \lambda 1$.[2]

We seek an expression for the Green's function for the operator $L - \lambda 1$. Let $Y_\nu = Y_\nu(x,\lambda)$ be a fundamental system of solutions of the equation $l(Y) = \lambda Y$. We choose this system so that the $Y_\nu(x,\lambda)$ are integral, analytic, matrix-functions of the parameter λ (it is always possible to do this). We write $|W|$ for the determinant of the matrix

$$W = \begin{bmatrix} Y_1^{(n-1)} & & Y_n^{(n-1)} \\ Y_1^{(n-2)} & \cdots & Y_n^{(n-2)} \\ \cdot & & \cdot \\ Y_1 & & Y_n \end{bmatrix}, \tag{2}$$

and denote by V_ν, $\nu = 1,2,\dots, n$, the mth-order matrices consisting of the cofactors of the elements Y_ν in the determinant $|W|$. Further, we let W_ν, $\nu = 1,2,\dots,n$ denote the transposed matrices of V_ν.

We put

$$Z_\nu = \frac{1}{|W|} W_\nu, \tag{3}$$

$$g(x,\xi) = \begin{cases} \frac{1}{2}\sum_{\nu=1}^{n} Y_\nu(x)Z_\nu(\xi) \text{ for } \xi < x, \\ -\frac{1}{2}\sum_{\nu=1}^{n} Y_\nu(x)Z_\nu(\xi) \text{ for } \xi > x, \end{cases} \tag{4}$$

$$U = \begin{bmatrix} U_1(Y_1) & \cdots & U_1(Y_n) \\ \cdot & & \\ U_n(Y_1) & \cdots & U_n(Y_n) \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} W_{11} & \cdots & W_{1n} \\ & & \\ W_{n1} & \cdots & W_{nn} \end{bmatrix}.$$

Here the $W_{j\nu}$ are $m \times m$ matrices.

---

[1] For a fixed x and ξ, $P(x,\xi)f(\xi)$ means the result of applying the operator $P(x,\xi)$ to the vector $f(\xi)$.

[2] In this section 7.2, the author assumes that λ is *not* an eigenvalue.

I. *The Green's function $G(x,\xi,\lambda)$ for the operator $L - \lambda 1$ is given by the formula*

$$G(x,\xi,\lambda) = g(x,\xi,\lambda) - \sum_{j,\nu=1}^{n} Y_j(x) W_{j\nu} U_\nu(g). \tag{5}$$

The proof of formula (5) is similar to the proof of formula (34) in §3.7. If we apply the method of variation of the parameters to the equation $l(y) - \lambda y = f$, we obtain for its solution y the formula

$$y = \sum_{\nu=1}^{n} Y_\nu(x) c_\nu + \int_a^b g(x,\xi,\lambda) f(\xi) d\xi, \tag{6}$$

the analogue of formula (31) in §3.7; the c_ν here, of course, are constant vectors. By substituting this expression for y into the boundary conditions $U_\nu(y) = 0, \nu = 1,2,\ldots,n$, we find these vectors c_n; then by substituting the expressions for c_ν into (6), we arrive at formula (5).

If we write Δ for the determinant of the matrix U, then

$$W_{j\nu} = \frac{1}{\Delta} V_{j\nu}, \tag{7}$$

where $V_{j\nu}$ is the transpose of that mth-order matrix consisting of the cofactor of the element $U_\nu(Y_j)$ in the determinant Δ. Formula (5) can then be rewritten as

$$G(x,\xi,\lambda) = g(x,\xi,\lambda) - \frac{1}{\Delta} \sum_{j,\nu=1}^{n} Y_j(x) V_{j\nu} U_\nu(g). \tag{8}$$

3. The Analytic Nature of the Green's Function for the Operator $L - \lambda 1$

The functions $Y_\mu(x)$ and consequently also $g(x,\xi,\lambda)$, $V_{\mu\nu}$, as well as $U_\nu(g)$ and Δ are integral, analytic functions of the parameter λ. Hence it follows from (8) that:

II. *The Green's function for the operator L is a meromorphic matrix-function of the parameter λ, and only eigenvalues of the operator L can be poles of this function.*

We want to look at the case when λ_0 is a simple zero of the function Δ in more detail.

We assume that the adjoint operator exists as a differential operator of the type under consideration here, and let $z(x)$ be one of its eigenfunctions corresponding to the eigenvalue $\bar{\lambda}_0$; we denote by $y(x)z^*(\xi)$ the matrix composed of the elements $y_k(x)\overline{z_j(\xi)}$. Using the result in §7.2, and repeating the argument in §3.8, we can deduce:

III. *At a simple zero* λ_0 *of the function* $\Delta(\lambda)$,

$$G(x,\xi,\lambda) = -\frac{y(x)z^*(\xi)}{(\lambda-\lambda_0)\int_a^b (y,z)dx} + G_1(x,\xi,\lambda), \tag{9}$$

where $G_1(x,\xi,\lambda)$ *is regular in the neighbourhood of the point* λ_0. *If, in particular,*

$$\int_a^b (y,z)dx = 1,$$

then

$$G(x,\xi,\lambda) = -\frac{y(x)z^*(\xi)}{\lambda-\lambda_0} + G_1(x,\xi,\lambda). \tag{10}$$

For differential operators of the first order in the space of vector functions, the Green's function was the subject of investigation in papers by Bunitski [13], Bôcher [11], Birkhoff and Langer [6], and Bliss [8].

§8. *Asymptotic Behaviour of the Eigenvalues of a Differential Operator*

1. Statement of the Problem

In §4 we examined the asymptotic behaviour of eigenvalues and eigenfunctions of a differential operator in the space of scalar functions. The results obtained enabled us to derive in §5 some general theorems on expansions in terms of eigenfunctions. In this section we shall be concerned with the analogous problem, the asymptotic behaviour of the eigenvalues and eigenfunctions of a differential operator, in the space of vector functions.

To solve this problem, we first investigate the asymptotic behaviour of solutions of the matrix equation

$$l(Y) + \rho^n Y = 0 \tag{1}$$

for large values of $|\rho|$. The decomposition of the complex ρ-plane into the domains S_k or T_k (see §4.2) plays an important part here.

For simplicity of exposition we discuss the case where $P_0(x) \equiv 1$, and therefore take as our differential expression

$$l(Y) = Y^{(n)} + P_1 Y^{(n-1)} + \dots + P_n Y. \tag{2}$$

(The possible generalizations to the case $P_0(x) \neq 1$ still remain open for consideration). Further, without loss of generality, we may assume that $P_1(x) \equiv 0$. For, otherwise, we can apply to equation (1) the substitution $Y = U\hat{Y}$, where U is a solution of the matrix-equation

$$U' + \frac{1}{n}P_1 U = 0$$

satisfying the condition

$$\det U \neq 0.$$

We then obtain for $\hat{Y}$ an equation of the form

$$l(\hat{Y})+\rho^n\hat{Y}=0$$

in which the coefficient of $\hat{Y}^{(n-1)}$ vanishes.

Finally, we can assume $a=0$ and $b=1$, and so we are considering the interval $[0,1]$. (See §4.1).

2. Asymptotic Behaviour of the Solution of the Matrix Equation $l(Y)+\rho^n Y=0$ for large $|\rho|$

Let then

$$l(Y)=Y^{(n)}+P_2Y^{(n-2)}+\dots+P_nY. \tag{3}$$

We consider the matrix-equation

$$l(Y)+\rho^n Y=0. \tag{4}$$

By applying to this equation essentially the same considerations as in §4.5 we obtain the following theorem.

THEOREM 1

If the matrix-functions $P_2,\dots,P_n$ *are continuous in the interval* $[0,1]$,[1] *then the matrix-equation*

$$Y^{(n)}+P_2Y^{(n-2)}+\dots+P_nY+\rho^nY=0$$

has in each domain T *of the complex* ρ*-plane* n *linearly independent solutions* $Y_1,Y_2,\dots,Y_n$ *which are analytic in* $\rho\in T$ *and which for sufficiently large* $|\rho|$, $\rho\in T$, *satisfy the relations*[2]

$$\left.\begin{aligned} Y_k&=e^{\rho\omega_kx}\left[1+O\left(\frac{1}{\rho}\right)\right],\\ \frac{dY_k}{dx}&=\rho e^{\rho\omega_kx}\left[\omega_k1+O\left(\frac{1}{\rho}\right)\right],\\ &\cdot\quad\cdot\\ \frac{d^{n-1}Y_k}{dx^{n-1}}&=\rho^{n-1}e^{\rho\omega_kx}\left[\omega_k^{n-1}1+O\left(\frac{1}{\rho}\right)\right].\end{aligned}\right\} \tag{5}$$

Here $O\left(\frac{1}{\rho}\right)$ is a matrix of the form $\frac{A(x,\rho)}{\rho}$ with $A(x,\rho)$ uniformly bounded:

$$|A_{ij}(x,\rho)|\leqslant M \text{ for } |\rho|\geqslant R,\ 0\leqslant x\leqslant 1,$$

where M and R are independent of x.

---

[1] See the footnote, p. 48.

[2] Here 1 stands for the nth-order unit matrix.

If the functions $P_k(x)$ are not only continuous but also sufficiently often differentiable, then the asymptotic formulae (5) can be made more precise, and formulae of the form

$$Y_k(x) = e^{\rho\omega_k x}\left[1 + \frac{Y_{k1}(x)}{\rho} + \frac{Y_{k2}(x)}{\rho^2} + \ldots + \frac{Y_{k\nu}(x)}{\rho^\nu} + O\left(\frac{1}{\rho^{\nu+1}}\right)\right]$$

can be obtained, with similar formulae for the derivatives $Y_k^{(\nu)}, \nu = 1,2,\ldots,n-1$. (See §4.6 for the details).

3. Normalization of the Boundary Conditions

A given differential operator is characterised by the boundary conditions $U_\nu(y) = 0$, $\nu = 1,2,\ldots,n$.

The number k is called the *order* of a form $U(y)$ if $U(y)$ contains at least one of the vectors $y_0^{(k)}$ and $y_1^{(k)}$ but does not contain the vectors $y_0^{(\nu)}$ or $y_1^{(\nu)}$ for $\nu > k$.

We consider forms $U(y)$ of order $(n - 1)$, if there are any; they have the form

$$U_\nu(y) = A_{\nu,n-1}y_0^{(n-1)} + B_{\nu,n-1}y_1^{(n-1)} + \ldots .$$

The rectangular matrix $[A_{\nu,n-1}, B_{\nu,n-1}]$ has m rows and $2m$ columns. The maximum number of linearly independent rows of $2m$ elements is, however, $2m$; if, then, we replace the rows in the forms of order $(n - 1)$ by linear combinations of these rows (if this process is necessary), we can arrange that not more than two forms of order $(n - 1)$ occur.

Continuing in the same way with the remaining forms, we can, after a finite number of such steps, reduce the boundary conditions to the form

$$U_\nu(y) = U_{\nu 0}(y) + U_{\nu 1}(y) = 0, \tag{6}$$

where

$$\left.\begin{aligned} U_{\nu 0}(y) &= A_\nu y_0^{(k_\nu)} + \sum_{j=0}^{k_\nu - 1} A_{\nu j} y_0^{(j)}, \\ U_{\nu 1}(y) &= B_\nu y_1^{(k_\nu)} + \sum_{j=0}^{k_\nu - 1} B_{\nu j} y_1^{(j)}, \end{aligned}\right\} \tag{7}$$

$$n - 1 \geqslant k_1 \geqslant k_2 \geqslant \ldots \geqslant k_n \geqslant 0,\ k_{\nu+2} > k_\nu,$$

and where, for each ν, $\nu = 1,2,\ldots,n$, at least one of the matrices A_ν, B_ν is different from the zero-matrix.

The operations just described are referred to as the *normalization* of the boundary conditions, and the finally resulting boundary conditions of the form (7) are called *normalized boundary conditions.*

4. Regular Boundary Conditions

We consider a fixed domain S_k, and number $\omega_1, \omega_2, \ldots, \omega_n$ so that, for $\rho \in S$,

$$\mathscr{R}(\rho\omega_1) \leqslant \mathscr{R}(\rho\omega_2) \leqslant \ldots \leqslant \mathscr{R}(\rho\omega_n).$$

The asymptotic formulae which follow are going to be derived for a particular class of boundary conditions which we shall call *regular*. The definition of regular boundary conditions depends on whether n is even or odd.

(a). *Suppose n is odd*; $n = 2\mu - 1$.

The normalized boundary conditions (6) *are said to be regular if both the numbers θ_0 and θ_m defined by the equation*

$\theta_0 + \theta_1 s + \ldots + \theta_m s^m =$

$$\begin{vmatrix} A_1\omega_1^{k_1} & A_1\omega_{\mu-1}^{k_1} & (A_1 + sB_1)\omega_\mu^{k_1} & B_1\omega_{\mu+1}^{k_1} & \ldots & B_1\omega_n^{k_1} \\ A_2\omega_1^{k_2} & A_2\omega_{\mu-1}^{k_2} & (A_2 + sB_2)\omega_\mu^{k_2} & B_2\omega_{\mu+1}^{k_2} & \ldots & B_2\omega_n^{k_2} \\ & & & \cdot & \ldots & \cdot \\ A_n\omega_1^{k_n} & A_n\omega_{\mu-1}^{k_n} & (A_n + sB_n)\omega_\mu^{k_n} & B_n\omega_{\mu+1}^{k_n} & \ldots & B_n\omega_n^{k_n} \end{vmatrix} \tag{8 a}$$

do not vanish.

(b). *Suppose n is even*; $n = 2\mu$.

The normalized boundary conditions (6) *are said to be regular if both the numbers θ_{-m} and θ_m defined by the equation*

$\theta_{-m}s^{-m} + \theta_{-m+1}s^{-m+1} + \ldots + \theta_m s^m =$

$$\begin{vmatrix} A_1\omega_1^{k_1} & \ldots & A_1\omega_{\mu-1}^{k_1} & (A_1 + sB_1)\omega_\mu^{k_1} & \left(A_1 + \frac{1}{s}B_1\right)\omega_{\mu+1}^{k_1} & B_1\omega_{\mu+2}^{k_1} & \ldots & B_1\omega_n^{k_1} \\ A_2\omega_1^{k_2} & \ldots & A_2\omega_{\mu-1}^{k_2} & (A_2 + sB_2)\omega_\mu^{k_2} & \left(A_2 + \frac{1}{s}B_2\right)\omega_{\mu+1}^{k_2} & B_2\omega_{\mu+2}^{k_2} & \ldots & B_2\omega_n^{k_2} \\ \cdot & \ldots & \cdot & \cdot & \cdot & \cdot & \ldots & \cdot \\ A_n\omega_1^{k_n} & \ldots & A_n\omega_{\mu-1}^{k_n} & (A_n + sB_n)\omega_\mu^{k_n} & \left(A_n + \frac{1}{s}B_n\right)\omega_{\mu+1}^{k_n} & B_n\omega_{\mu+2}^{k_n} & \ldots & B_n\omega_n^{k_n} \end{vmatrix} \tag{8 b}$$

do not vanish.

We can see at once that regularity so defined does not depend on the particular domain S_k selected. For, the numbers θ_0 and θ_m (or θ_{-m} and θ_m) are determinants, which are obtained from those used in §4 to define the numbers θ, merely by replacing α_k and β_k by the matrices A_k and B_k. Hence precisely the same laws hold for their transformation due to a change in domain S_k; in particular, for odd n, they are multiplied by a factor of modulus 1. For n even, the whole of the calculation carried out in §4.8 can be analogously repeated.

It follows, therefore, that here too the roots of the equation

$$\theta_{-m}s^{-m} + \theta_{-m+1}s^{-m+1} + \ldots + \theta_m s^m = 0$$

transform into their reciprocals when the domain is changed from S_0 to S_1 or S_{2n-1}. For odd n, there are two classes of equations

$$\theta_0 + \theta_1 s + \ldots + \theta_m s^m = 0,$$

with the same roots in each case in each class. For S_k with k odd, the roots are $\xi_1^{(1)},\ldots,\xi_m^{(1)}$; and for S_k with k even, the roots are $\xi_1^{(2)},\ldots,\xi_m^{(2)}$.

5. Asymptotic Behaviour of the Eigenvalues

In questions concerning the asymptotic behaviour of the eigenvalues, the equation

$$\theta_0 + \theta_1 s + \ldots + \theta_m s^m = 0 \tag{9 a}$$

for odd n, and the equation

$$\theta_{-m}s^{-m} + \theta_{-m+1}s^{-m+1} + \ldots + \theta_m s^m = 0 \tag{9 b}$$

for even n, both play an important part; the numbers θ_j here are the numbers defined in the previous subsection 4.

If the boundary conditions are regular, these equations are respectively of order m and $2m$, and all their roots are different from zero.

In the following theorem we assume (though for brevity we do not mention this again in the hypothesis) that the coefficients of the differential expression considered are continuous matrix-functions in the interval [0,1]. It should, however, be particularly mentioned that all the results of this subsection would still hold if we required only that these coefficients be arbitrary, summable, matrix-functions in the interval [0,1].

THEOREM 2

Let L be a differential operator of the nth order, defined in the interval [0,1], *whose differential expression contains no derivative of the* (n — 1)*th order, and whose boundary conditions are regular.*

(1). If n is odd, *then to each simple root* $\xi_j^{(1)}$ *corresponds a sequence* $\lambda_{kj}^{(i)}$, *and to each simple root* $\xi_j^{(2)}$ *corresponds a sequence* $\lambda_{kj}^{(2)}$, *of eigenvalues of the operator L, and*

$$\left.\begin{aligned}
\lambda_{kj}^{(1)} &= (\mp 2k\pi i)^n\left[1 \mp \frac{n \ln_0 \xi_j^{(1)}}{2k\pi i} + O\left(\frac{1}{k^2}\right)\right]\\
\lambda_{kj}^{(2)} &= (\pm 2k\pi i)^n\left[1 \pm \frac{n \ln_0 \xi_j^{(2)}}{2k\pi i} + O\left(\frac{1}{k^2}\right)\right]
\end{aligned}\right\} k = N,(N+1),\ldots, \tag{10 a}$$

where the upper or lower signs hold according as $n = 4\nu + 1$ *or* $n = 4\nu - 1$.

To an r-fold root $\xi_j^{(1)}$ *or* $\xi_j^{(2)}$, *on the other hand, correspond in general r sequences of eigenvalues* $\lambda_{kj}^{(1)}$ *or* $\lambda_{kj}^{(2)}$ *which for large values of k satisfy the following formulae:*

$$\left.\begin{aligned} \lambda_{kj}^{(1)} &= (\mp 2k\pi i)^n\left[1 \mp \frac{n \ln_0 \xi_j^{(1)}}{2k\pi i} + O\left(\frac{1}{k^{1+1/r}}\right)\right] \\ \lambda_{kj}^{(2)} &= (\pm 2k\pi i)^n\left[1 \pm \frac{n \ln_0 \xi_j^{(2)}}{2k\pi i} + O\left(\frac{1}{k^{1+1/r}}\right)\right] \end{aligned}\right\} \begin{aligned} &j = j_0, (j_0+1), \ldots, (j_0+r-1), \\ &k = N(N+1), \ldots . \end{aligned} \tag{11a}$$

Here it may be taken that $\xi_{j_0}^{(\nu)} = \xi_{j_0+1}^{(\nu)} = \ldots = \xi_{j_0+r-1}^{(\nu)}$.

The same rule for upper or lower signs holds as for formulae (10a). *These eigenvalues* $\lambda_{kj}^{(\nu)}, j = j_0, (j_0+1), \ldots, (j_0+r-1)$, *may coincide for certain j and k to give multiple eigenvalues.*

(2). If n is even, *then to each simple root* ξ *of the equation* (9b) *for the domain* S_0 *corresponds a sequence* λ_k *of eigenvalues of the operator L, and*

$$\lambda_k = (2k\pi i)^n\left[1 \mp \frac{n \ln_0 \xi}{2k\pi i} + O\left(\frac{1}{k^2}\right)\right], \quad k = N, (N+1), \ldots, \tag{10b}$$

where the upper or lower signs hold according as $n = 4\nu$ *or* $n = 4\nu + 2$.

To each multiple zero ξ *of equation* (9b), *with multiplicity r, correspond r sequences of eigenvalues* λ_{kj} *of the operator L, and*

$$\lambda_{kj} = (2k\pi i)^n\left[1 \mp \frac{n \ln_0 \xi}{2k\pi i} + O\left(\frac{1}{k^{1+1/r}}\right)\right], j = 1, 2, \ldots, r; k = N, (N+1), \ldots, \tag{11b}$$

and again the sign is $-$ *or* $+$ *according as* $n = 4\nu$ *or* $n = 4\nu + 2$.

For sufficiently large $|\lambda|$, *the operator L has no other eigenvalues and the multiplicity of any particular eigenvalue* λ *is equal to the number of eigenvalues from the above sequences which happen to coincide there, i.e. the multiplicity is not greater than the multiplicity of the corresponding root* ξ *of equation* (9).

Proof. First let n be odd, say $n = 2\mu - 1$.

We consider a fixed domain T; let the numbers $\omega_1, \omega_2, \ldots, \omega_n$ be numbered in sequence so that, for $\rho \in T$,

$$\mathscr{R}((\rho + c)\omega_1) \leqslant \mathscr{R}((\rho + c)\omega_2) \leqslant \ldots \leqslant \mathscr{R}((\rho + c)\omega_n).$$

By applying formula (5) of subsection 2, and repeating the argument in the proof of Theorem 2 in §4.9, we obtain

$$\left.\begin{aligned} U_\nu(Y_j) &= (\rho\omega_j)^{k_\nu}[A_\nu] && \text{for } j < \mu, \\ U_\nu(Y_j) &= (\rho\omega_j)^{k_\nu}e^{\rho\omega_j}[B_\nu] && \text{for } j > \mu, \\ U_\nu(Y_\mu) &= (\rho\omega_\mu)^{k_\nu}\{[A_\nu] + e^{\rho\omega_\mu}[B_\nu]\}. && \end{aligned}\right\} \tag{12}$$

We substitute all these expressions into the equation $\Delta = 0$; it follows from the definition (8 a) of the numbers θ_k that, after removing the factors $\rho^{k_1},\rho^{k_2},\ldots,\rho^{k_n},e^{\rho\omega_{\mu+1}},\ldots,e^{\rho\omega_\mu}$, this equation takes the form

$$[\theta_0] + [\theta_1]e^{\omega_\mu\rho} + [\theta_2]e^{2\omega_\mu\rho} + \ldots + [\theta_n]e^{n\omega_\mu\rho} = 0.$$

Hence, and by arguments similar to those used in proving Theorem 2 of §4.9, follow all the assertions of the Theorem, for n odd.

The case of n even is treated in similar fashion.

By using Theorems 1 and 2 we can obtain asymptotic formulae for the eigenfunctions, but we shall not go into this question in detail.

§9. *Expansion in Terms of Eigenfunctions of a Differential Operator*

1. The Case of a Self-Adjoint Differential Operator

Let L be a self-adjoint differential operator in the space of vector-functions. By repeating the argument of §5.2, we obtain the following theorem.

THEOREM 1

Any vector-function in the domain of definition of a self-adjoint operator L can be expanded in a uniformly convergent, generalized Fourier series in terms of the eigenfunctions of L.

If $y_1(x),y_2(x),\ldots$ is an orthonormal system of eigenfunctions of the operator L, so that all eigenfunctions of L for arbitrary eigenvalues can be expressed as linear combinations of these vectors, then the expansion referred to in Theorem 1 has the form

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) \tag{1}$$

with

$$c_n = \int_b^a (f(x),y_n(x))dx. \tag{2}$$

We recall once more that $f(x)$ and $y_n(x)$ are vector-functions; if, then, we put

$$f(x) = \{f_1(x),f_2(x),\ldots,f_m(x)\},$$
$$y_n(x) = \{y_{n1}(x),y_{n2}(x),\ldots,y_{nm}(x)\}, \quad n = 1,2,3,\ldots,$$

then the representation (1) is equivalent to the system of expansions in uniformly convergent series

$$f_j(x) = \sum_{n=1}^{\infty} c_n y_{nj}(x), \quad j = 1,2,\ldots,m,$$

the same coefficients c_n occurring in each of the series.

It follows from Theorem 1 that the functions $y_1(x), y_2(x), y_3(x), \ldots$ form a complete orthonormal system in the space $L^2_m(a,b)$ of all measurable vector functions $f(x)$ which satisfy the condition

$$\int_a^b |f(x)|^2 dx < +\infty.$$

[$|f(x)|^2$ means here the scalar product of the vector $f(x)$ with itself.]

It follows that

$$\int_a^b |f(x)|^2 dx = \sum_{n=1}^{\infty} |c_n|^2, \tag{3}$$

with

$$c_n = \int_a^b (f(x), y_n(x)) dx.$$

2. Expansion in terms of the Eigenfunctions of a Differential Operator Generated by Regular Boundary Conditions

The considerations in §9.1 are not applicable if L is not a self-adjoint operator.

So we now make use of a different method which depends on the analytic properties of the Green's function for the operator $L - \lambda 1$ and on the asymptotic formula obtained in §8. Here we can assume that the boundary conditions which generate L are regular; further, without loss of generality, we may assume that the boundary-value problem $Ly = 0$ has only the trivial solution. Then L has a Green's function $G(x,\xi)$. As in §5.3, we consider a sequence of circles Γ_k, $k = 1,2,\ldots$, with centre at the origin, and having the following properties:

1°. The radius R_k of the circle Γ_k increases without limit as $k \to \infty$.

2°. There is a positive number δ such that all eigenvalues of the operator L have a distance $\geqslant \delta$ from each of the circles Γ_k.

By virtue of the asymptotic properties of these eigenvalues proved in §8.5, such circles do exist.

Let $G(x,\xi,\lambda)$ be the Green's function for the operator $L - \lambda 1$, and, in particular, let $G(x,\xi,0) = G(x,\xi)$ be the Green's function for the operator L. We consider the integral

$$I_k = \frac{1}{2\pi i} \oint_{\Gamma_k} \frac{G(x,\xi,\lambda)}{\lambda} d\lambda. \tag{4}$$

LEMMA

On the circles Γ_k the function $G(x,\xi,\lambda)$ satisfies the inequality

$$|G_{jk}(x,\xi,\lambda)| \leqslant \frac{M}{|\lambda|^{1-1/n}}, \tag{5}$$

where M is a fixed constant, uniformly for $k = 1,2,3,\ldots$ and $a \leqslant x,\xi \leqslant b$.

Proof. If we put $\lambda = -\rho^n$, then, for a suitable choice of arg ρ, the circle Γ_k is transformed into a circular arc γ_k subtending an angle $2\pi/n$ at the centre, and γ_k goes through two neighbouring domains S_1, S_2 of the complex ρ-plane. We want to estimate the value of the function $G(x,\xi,\lambda)$ on the arc γ_k.

Let the numbers $\omega_1,\omega_2,\ldots,\omega_n$ be numbered in sequence so that, for $\rho \in S_1$,

$$\mathscr{R}(\rho\omega_1) \leqslant \mathscr{R}(\rho\omega_2) \leqslant \ldots \leqslant \mathscr{R}(\rho\omega_n).$$

We use the formula for $G(x,\xi,\lambda)$ from §7.2, and substitute therein the asymptotic expressions for $Y_j(x,\lambda)$ found in §8.2. We get

$$Z_\nu(\xi) = e^{-\rho\omega_\nu\xi}\frac{1}{n\rho^{n-1}}[-\omega_\nu 1], \tag{6}$$

and so, by (4) in §7.2,

$$g(x,\xi) = \mp\frac{1}{2n\rho^{n-1}}\left(\sum_{\nu=1}^{n} e^{\rho\omega_\nu(x-\xi)}\right)[1], \tag{7}$$

where the $-$ sign is to be taken if $\xi < x$, and the $+$ sign if $\xi > x$.

For the rest of the proof it is best to deal with n odd and n even separately.

First let n be odd; $n = 2\mu - 1$. Then, for $\rho \in S_1$, we have

$$\left.\begin{aligned} &\mathscr{R}(\rho\omega_1) \leqslant 0,\ldots, \mathscr{R}(\rho\omega_{\mu-1}) \leqslant 0,\\ &\mathscr{R}(\rho\omega_{\mu+1}) \geqslant 0,\ldots, \mathscr{R}(\rho\omega_n) \geqslant 0. \end{aligned}\right\} \tag{8}$$

We denote by γ_k' that part of the arc γ_k which lies in S_1 and on which $\mathscr{R}(\rho\omega_\mu) \geqslant 0$, and by γ_k'' the part of γ_k which lies in S_1 and on which $\mathscr{R}(\rho\omega_\mu) \leqslant 0$. We want to estimate the value of the function $G(x,\xi,\lambda)$ on the arc γ_k'. By using formula (12) of §8.5, it can easily be shown that, on γ_k',

$$Y_j(x)W_{j\nu} = O\left(\frac{1}{\rho^{k_\nu}}\right). \tag{9}$$

To obtain an estimate of $G(x,\xi,\lambda)$ on γ_k', we first transform the expression for $G(x,\xi,\lambda)$ in formula (5) of §7.2. By virtue of formula (4) in §7.2, for $g(x,\xi)$ we have the equations

$$g(0,\xi) = -\tfrac{1}{2}\sum_{\alpha=1}^{n} Y_\alpha(0)Z_\alpha(\xi), \quad g(1,\xi) = \tfrac{1}{2}\sum_{\alpha=1}^{n} Y_\alpha(1)Z_\alpha(\xi);$$

consequently

$$U_\nu(g) = \tfrac{1}{2}\sum_{\alpha=1}^{n} [U_{\nu 1}(Y_\alpha) - U_{\nu 0}(Y_\alpha)]Z_\alpha(\xi).$$

We write this expression in the form

$$U_\nu(g) = \tfrac{1}{2}\sum_{\alpha=1}^{\mu}[-U_\nu(Y_\alpha) + 2U_{\nu1}(Y_\alpha)]Z_\alpha(\xi) + \tfrac{1}{2}\sum_{\alpha=\mu+1}^{n}[U_\nu(Y_\alpha) - 2U_{\nu1}(Y_\alpha)]Z_\alpha(\xi)$$

$$= -\tfrac{1}{2}\sum_{\alpha=1}^{\mu} U_\nu(Y_\alpha)Z_\alpha(\xi) + \sum_{\alpha=1}^{\mu} U_{\nu1}(Y_\alpha)Z_\alpha(\xi)$$

$$+ \tfrac{1}{2}\sum_{\alpha=\mu+1}^{n} U_\nu(Y_\alpha)Z_\alpha(\xi) - \sum_{\alpha=\mu+1}^{n} U_{\nu0}(Y_\alpha)Z_\alpha(\xi). \quad (10)$$

By (6), we have, however, for $\alpha \leqslant \mu$,

$$U_{\nu1}(Y_\alpha)Z_\alpha(\xi) = -\frac{\rho^{k_\nu}}{n\rho^{n-1}}\,\omega_\alpha^{k_\nu+1}e^{\rho\omega_\alpha(1-\xi)}[B_\nu];$$

hence, and by (8), it follows that

$$U_{\nu1}(Y_\alpha)Z_\alpha(\xi) = O\left(\frac{1}{\rho^{n-k_\nu-1}}\right).$$

Similarly, for $\alpha > \mu$, we get

$$U_{\nu0}(Y_\alpha)Z_\alpha(\xi) = O\left(\frac{1}{\rho^{n-k_\nu-1}}\right).$$

So (10) can be written in the form

$$U_\nu(g) = -\tfrac{1}{2}\sum_{\alpha=1}^{\mu} U_\nu(Y_\alpha)Z_\alpha(\xi) + \tfrac{1}{2}\sum_{\alpha=\mu+1}^{n} U_\nu(Y_\alpha)Z_\alpha(\xi) + O\left(\frac{1}{\rho^{n-k_\nu-1}}\right). \quad (11)$$

If this representation of $U_\nu(g)$ is now substituted into formula (5), §7.2, for $G(x,\xi,\lambda)$, we obtain

$$G(x,\xi,\lambda) = g(x,\xi,\lambda) + \tfrac{1}{2}\sum_{j=1}^{\mu} Y_j(x)Z_j(\xi) - \tfrac{1}{2}\sum_{j=\mu+1}^{n} Y_j(x)Z_j(\xi) + O\left(\frac{1}{\rho^{n-1}}\right), \quad (12)$$

since, by definition of the inverse matrix,

$$\sum_{\nu=1}^{n} W_{j\nu}U_\nu(Y_\alpha) = \delta_{j\alpha}1,$$

and further, by (9),

$$Y_j(x)W_{j\nu}.O\left(\frac{1}{\rho^{n-k_\nu-1}}\right) = O\left(\frac{1}{\rho^{n-1}}\right).$$

Combining (12) with (4) from §7.2, we arrive at the formulae

$$G(x,\xi,\lambda) = \begin{cases} \displaystyle\sum_{j=1}^{\mu} Y_j(x)Z_j(\xi) + O\left(\frac{1}{\rho^{n-1}}\right) & \text{for } \xi < x, \\ \displaystyle -\sum_{j=\mu+1}^{n} Y_j(x)Z_j(\xi) + O\left(\frac{1}{\rho^{n-1}}\right) & \text{for } \xi > x. \end{cases} \quad (13)$$

By (8) and the asymptotic formulae for $Y_j(x)$ and $Z_j(\xi)$, it follows that on the arc γ_k' for $\xi < x$,

$$\sum_{j=1}^{\mu} Y_j(x)Z_j(\xi) = -\frac{1}{n\rho^{n-1}}\sum_{j=1}^{\mu}[\omega_j]e^{\rho\omega_j(x-\xi)} = O\left(\frac{1}{\rho^{n-1}}\right),$$

and for $\xi > x$,

$$\sum_{j=\mu+1}^{n} Y_j(x)Z_j(\xi) = -\frac{1}{n\rho^{n-1}}\sum_{j=\mu+1}^{n}[\omega_j]e^{\rho\omega_j(x-\xi)} = O\left(\frac{1}{\rho^{n-1}}\right).$$

In conjunction with (13) this gives the desired relation

$$G(x,\xi,\lambda) = O\left(\frac{1}{\rho^{n-1}}\right)$$

on the arc γ_k'.

Similarly we can verify this relation on the arc γ_k'', and on that part of the arc γ_k which lies in the domain S_2.

This proves the lemma for the case of n odd.

For n even, the proof runs on exactly similar lines.

Using these Lemmata, the following theorems may be deduced exactly as in §5.3:

THEOREM 2

The Green's function $G(x,\xi)$ of a differential operator which is generated by regular boundary conditions can be represented as a series

$$G(x,\xi) = -\sum_{\nu=1}^{\infty}\frac{H_\nu(x,\xi)}{\lambda_\nu}$$

which, at least for a certain definite order of its terms, converges uniformly. Here $H_\nu(x,\xi)$ denotes the residue of the function $G(x,\xi,\lambda)$ at the pole λ_ν.

THEOREM 3

If all the eigenvalues of an operator L generated by regular boundary conditions are simple zeros of the function $\Delta(\lambda)$, then its Green's function $G(x,\xi)$ can be represented as a series

$$G(x,\xi) = \sum_{\nu=1}^{\infty}\frac{y_\nu(x)z_\nu^*(\xi)}{\lambda_\nu}$$

which, at least for a certain definite order of its terms, converges uniformly. Here $y_\nu(x)$ and $z_\nu(x)$ are the eigenfunctions, of the operators L and L^* respectively, associated with the eigenvalues λ_ν and $\bar{\lambda}_\nu$; and those eigenfunctions may both be so normalized that

$$\int_a^b (y_\nu(x),z_\nu(x))dx = 1.$$

THEOREM 4

Let L be an operator which is generated by regular conditions and of which all the eigenvalues are simple zeros of the function Δ. *Then any function* $f(x)$ *in the domain of definition of the operator L can be expanded as a series in eigenfunctions of L:*

$$f(x) = \sum_{\nu=1}^{\infty} c_\nu y_\nu(x), \text{ where } c_\nu = \int_a^b (f(\xi), z_\nu(\xi))d\xi.$$

The series converges uniformly, at least for one particular order of its terms. Here $y_\nu(x)$ *and* $z_\nu(x)$ *are eigenfunctions of L and L*, for the eigenvalues* λ_ν *and* $\bar{\lambda}_\nu$ *respectively; these eigenfunctions may be normalized so that*

$$\int_a^b (y_\nu(x), z_\nu(x))dx = 1.$$

For simplicity we have restricted ourselves to the ordinary eigenvalue-problem and have not discussed the generalized problem $Ly = \lambda My$ at all, nor the problem $Ly = \lambda B(x)y$ in particular. For the latter problem, if certain restrictions are imposed on $B(x)$, it can be assumed that $B(x)$ is a diagonal matrix which has $\mu_1(x), \mu_2(x), \ldots, \mu_n(x)$ as its diagonal elements, the $\mu_\nu(x)$ being continuous and all different (see §6.8). We can obtain, for the solution of the corresponding differential equation and for the eigenvalues, certain asymptotic formula in which

$$\exp\left\{\rho \int_a^b \mu_j(t)dt\right\}$$

takes over the rôle played by $\exp\{\rho\omega_j(x - a)\}$ previously. These formulae can be used to obtain expansions in terms of eigenfunctions. It may be mentioned that no one has investigated up to now either this case or the general case $Ly = \lambda My$ in any detail (except for the special cases $n = 1$ or $m = 1$).

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